# New Constructiton for VMT labelings for products of graphs

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**Abstract.** A vertex-magic total labeling of a graph $G(V, E)$ is defined as one-to-one mapping from the set of integers $\{1, 2, ..., |V| + |E|\}$ to $V \cup E$ with the property that the sum of the label of a vertex and the labels of all edges incident to this vertex is the same constant for all vertices of the graph.
In the article we present a method for constructing vertex magic total labelings of Cartesian products of odd cycles and $K_5$. We also present some new techniques for VMT labelings of products of even cycles and some regular vertex magic total graphs with additional properties.

**Key Words:** graph, labeling, VMT, Cartesian product

## 1 Motivation

Let $G(V, E)$ be a graph. A labeling of $G$ is assigning labels to the vertices, edges or both vertices and edges. In most applications label are positive (or nonnegative) integers, though in general real numbers could be used.

The notion of magic squares can be naturally extended to graphs. We want the sum of labels related to an edge or a vertex be constant all over the graph. If the sum of labels of an edge and both end vertices does not depend on the edge, we call the labeling edge-magic type labeling. If the sum of labels of a vertex and all incident edges is constant, we call the labeling vertex-magic type labeling.

In general graph labelings prove to be useful in a number of applications. For example graceful labeling is used in graph decompositions. Magic type labelings play a role if a check sum is required or a look up table has to be avoided. A simple graph can represent a network with nodes and links with addresses (labels) assigned to both links and nodes. If addresses form an edge-magic labeling, knowing addresses of two vertices is enough to find the address of their link without a look up table simply by subtracting addresses from the magic constant. Another application of magic type labelings is in radar pulse codes where an optimal transmitted pulse-train can be obtained.

My studies are focused on vertex magic total labelings.

**Definition 1.** *Let $G$ be a graph with vertex set $V$ and edge set $E$. Let $v = |V|$ and $e = |E|$. A one-to-one mapping from $V \cup E$ to the set of integers $\{1, 2, \ldots, v + e\}$ is a* vertex magic total labeling *(VMT labeling) if there is a constant $h$ so that for every vertex $x$ is*

$$\lambda(x) + \sum \lambda(xy) = h$$

*where the sum is over all edges $xy$ incident to $x$.*
*The constant $h$ is the* magic constant *for $\lambda$.*

An example of VMT labeling is in the figure 1.

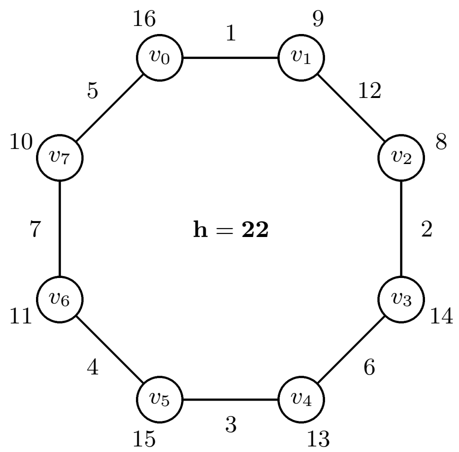


**Fig. 1.** VMT labeling for $C_8$

## 2 Summary of known results

There was a large number of articles published on magic-type graph labelings. Probably the best source of information is *Dynamic Survey of Graph Labeling* by Joseph Gallian [7]. The table 1 will appear in the next edition of the *Dynamic Survey*. Preparation of the summary tables was part of a summer project [8].

From the table it is apparent that the existence of VMT labeling is known for most basic families of graphs as complete graphs, bipartite graphs, cycles, etc. Notice there is only one general result concerning VMT labeling of copies of a VMT graph (see [11]).

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $C_n$ | VMT | [2] |
| $P_n$ | VMT | $n > 2$ [2] |
| $K_{m,m}$ | VMT | $m > 1$ [2][9] |
| $K_{m,m} - e$ | VMT | $m > 2$ [2] |
| $K_{m,n}$ | VMT | iff $\lvert m - n\rvert \leq 1$ [2] |
| $K_n$ | VMT | for $n$ odd [2] |
| | | for $n \equiv 2 \pmod 4$, $n > 2$ [9] |
| Petersen $P(n,k)$ | VMT | [1] |
| prisms $C_n \times P_2$ | VMT | |
| $W_n$ | VMT | iff $n \leq 11$ [2] |
| $F_n$ | VMT | iff $n \leq 10$ [2] |
| friendship graphs | VMT | iff # of triangles $\leq 3$ [2] |
| $G + H$ | VMT | $\lvert V(G)\rvert = \lvert V(H)\rvert$ |
| | | and $G \cup H$ is VMT [10] |
| unions of stars | VMT | [10] |
| Tree with $n$ | | |
| internal vertices | | |
| and more | not VMT | [10] |
| than $2n$ leaves | | [10] |
| $nG$ | VMT | $n$ odd, $G$ regular of even |
| | | degree, VMT [11] |
| $nG$ | VMT | $G$ is regular of odd |
| | | degree, VMT, but not $K_1$ [11] |
| $C_n \times C_{2m}$ | VMT | [3] |
| $K_5 \times C_{2n+1}$ | VMT | [4] |

**Table 1.** Summary of results on vertex magic total labelings.

## 3 VMT labeling for $K_5 \times C_n$ for $n$ odd

In the center of our attention are products of regular graphs. Based on antimagic labeling for cycles we have found a method for constructing VMT labeling for products of odd cycles (see [3]).

Generalizing methods used in the construction of VMT labeling for odd cycles the following result was obtained.

**Theorem 1.** *Let $n$ be odd. There exists a vertex magic total labeling of $K_5 \times C_n$.*

*Proof:* Let $K_5 \times C_n$ be the graph with vertex set $v_{i,j}$ for $i = 0, 1, \ldots, 4$, $j = 0, 1, \ldots, n-1$, and edge set $v_{i,j}v_{i,j+1}$, $v_{k,j}v_{l,j}$ for $k, l = 0, 1, \ldots, 4$, $k \neq l$ where indexes are taken mod $n$. Consider the following labeling

$$\lambda(v_{i,j}) = 2n(5-i) - 2j - 1$$

$$\lambda(v_{i,j}v_{i,j+1}) = \begin{cases} 2ni + 2n - j & \text{for } j \text{ even} \\ 2ni + (n+1) - j & \text{for } j \text{ odd} \end{cases}$$

$$\lambda(v_{0,j}v_{1,j}) = 10n + 9n + j + 1$$

$$\begin{aligned}
\lambda(v_{0,j}v_{2,j}) &= 10n + 2n + j + 1\\
\lambda(v_{0,j}v_{3,j}) &= 10n + 7n + j + 1\\
\lambda(v_{0,j}v_{4,j}) &= 10n + 4n + j + 1\\
\lambda(v_{1,j}v_{2,j}) &= 10n + 5n + j + 1\\
\lambda(v_{1,j}v_{3,j}) &= 10n + 0n + j + 1\\
\lambda(v_{1,j}v_{4,j}) &= 10n + 6n + j + 1\\
\lambda(v_{2,j}v_{3,j}) &= 10n + 8n + j + 1\\
\lambda(v_{2,j}v_{4,j}) &= 10n + 3n + j + 1\\
\lambda(v_{3,j}v_{4,j}) &= 10n + 1n + j + 1
\end{aligned} \tag{1}$$

for $i = 0, 1, \ldots, 4$, $j = 0, 1, \ldots, n-1$. See figure 2.

Expressing the sum of labels at vertices $v_{i,j}$ and all incident edges we get

$$\sum_{\substack{k=0\\k\neq j}}^{4} \lambda(v_{i,j}v_{k,j}) + \lambda(v_{i,j}) + \lambda(v_{i-1,j}v_{i,j}) + \lambda(v_{i,j}v_{i+1,j}) = 75n + 4 = h$$

for all vertices of $K_5 \times C_n$. Thus the labeling 1 is a VMT labeling of $G \times C_n$. $\square$

## 4 VMT labeling for Cartesian products of certain $(2r+1)$-regular graphs and even cycles

After careful examination of various constructions of VMT labelings for regular graphs, the following general result was obtained (mentioned also is [4]).

**Theorem 2.** *Let $n$ be even, let $r \geq 1$ be an integer. Let $G$ be a $(2r+1)$-regular VMT graph which can be factorized into two factors, a $(r+1)$-regular factor and a $r$-regular factor. There exists a vertex magic total labeling of $G \times C_n$.*

*Idea of the proof:* Since the entire proof of the theorem takes three pages, we will just introduce some basic notation and the construction of the labeling.

Let $G$ be a $(2r+1)$-regular VMT graph with vertex magic total labeling $f$ with magic constant $h$ which can be factorized into two factors, a $(r+1)$-regular factor $H_1$ and a $r$-regular factor $H_2$. Let $v$ denote the number of vertices of $G$ and $e$ the number of edges of $G$.

We will construct $n$ copies $G_i$ of graph $G$. Let us denote the vertices of $G$ as $v_1, v_2, \ldots, v_v$ and edges $e_1, e_2, \ldots, e_e$. In $G_i$ we denote the copies of vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,v}$ and edges $e_{i,1}, e_{i,2}, \ldots, e_{i,e}$.

Consider the following labeling:

$$\begin{aligned}
\lambda(v_{i,j}) &= \begin{cases} n(f(v_j)-1) + i + 1 & \text{if } i = 0\\ n(f(v_j)-1) + (n+1) - i & \text{if } i = 1, 2, \ldots, n-1 \end{cases}\\
\lambda(e_{i,k}) &= \begin{cases} n(f(e_k)-1) + n - i & \text{if } e_k \in H_1\\ n(f(e_k)-1) + i + 1 & \text{if } e_k \in H_2 \end{cases}\\
\lambda(v_{i,j}v_{i+1,j}) &= \begin{cases} n(v+e) + n(j-1) + i + 1 & \text{if } i \text{ is even}\\ n(v+e) + n(v-j) + n - i & \text{if } i \text{ is odd} \end{cases}
\end{aligned} \tag{2}$$

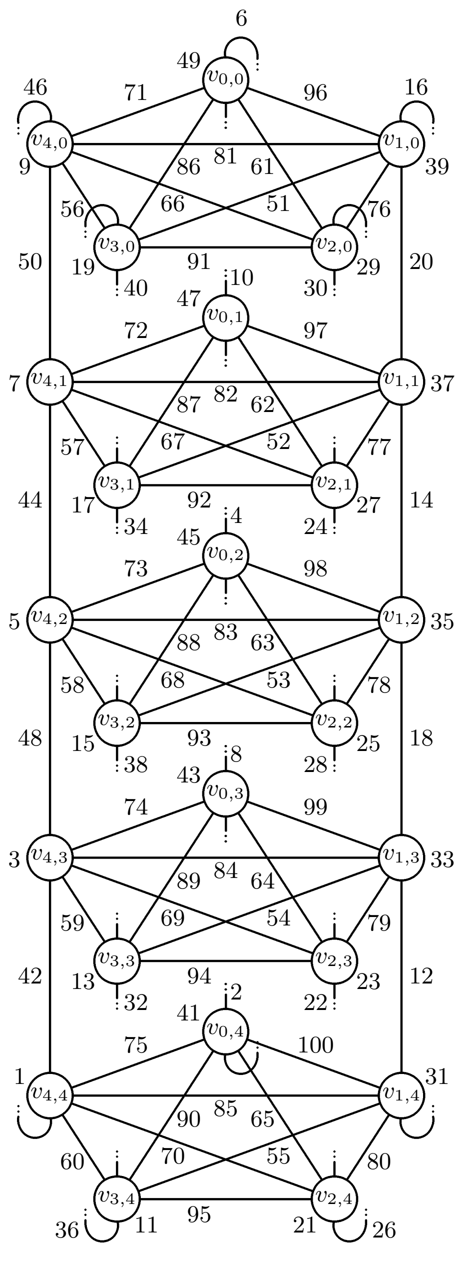

**Fig. 2.** VMT labeling of $K_5 \times C_5$

Evaluating the sum of labels at vertex $v_{i,j}$ and all incident edges we obtain

$$n(h + 3v + 2e - r - 1) + r + 2$$

which is a constant for all vertices of $G \times C_n$ for $n$ even. Thus the labeling 2 is a VMT labeling of $G \times C_n$. □

*Remark 1.* The result given in Theorem 2 is the most general result for VMT labelings so far. In comparison to known labelings for certain classes of graphs Theorem 2 enables to construct VMT labelings for products of *several* graph classes and even cycles, e.g., $K_{2n} \times C_{2m}$, $K_{2n+1,2n+1} \times C_{2m}$, $P(n,k) \times C_{2m}$, etc.

Another nice property of Theorem 2 is that the resulting graph *again* satisfies the condition and repetitive products can be obtained, as $(K_{2n} \times C_{2m}) \times C_{2k} \ldots$

*Remark 2.* Notice, that the classes of VMT graphs obtained in Theorem 1 are not among classes of VMT graphs constructed in Theorem 2, since the length of the cycle is odd in the first case and even in the second case.

## 5 Concluding remarks

The topic of VMT labelings for Cartesian products of graphs promises more interesting and even more general results to be found during my Ph.D. studies. Some other results were already published in conference proceedings [4] and [5].

## References


1. M. Bača, Mirka Miller and Slamin, Every generalized Petersen graphs has a vertex-magic total labeling, *preprint* (2002).
2. J.A. MacDougall, M. Miller, Slamin, W. D. Wallis, Vertex-Magic total labelings of graphs, *Util. Math*, **61** (1990) 3–21.
3. D. Fronček, P. Kovář, T. Kovářová, Vertex magic total labeling of Cartesian products of cycles, *Proceedings of Conference MIGHTY XXXVI*, Oshkosh (2003).
4. D. Fronček, P. Kovář, T. Kovářová, Vertex magic total labeling of Cartesian products of some vertex magic total regular graphs and odd cycles, *Proceedings of Conference MIGHTY XXXVI*, Oshkosh (2003).
5. D. Fronček, P. Kovář, T. Kovářová, Vertex magic total labeling of products of cycles, *to appear.*
6. P. Kovář, Vertex magic total labeling of Cartesian products of some regular VMT regular graphs and even cycles, *Proceedings of Conference MIGHTY XXXVII*, Valparaiso (2003).
7. Joseph A. Gallian, A Dynamic Survey of Graph Labeling *The electronic journal of combinatorics*, **5** (2002).
8. P. Kovář, T. Kovářová, Funding for project to include figures, tables and index to Dynamic Survey, summer project under prof. Gallian, *University of Minnesota Duluth*, (2003).
9. Y. Lin, M. Miller, Vertex-magic total labelings of complete graphs, *Bull. Inst. Combin. Appl*, (2002).
10. W.D. Wallis, Magic Graphs, *Birkhäuser* (2001).
11. W.D. Wallis, Vertex Magic Labelings of Multiple Graphs, *Congressus Numeratntium 152* (2001). pp. 81–83.