On Vertex-Magic Total labelings of products of cycles

Tereza Kovářová

Department of Applied Mathematics, FEI and Department of Mathematics and Descriptive Geometry, VŠB - Technick Univerzita Ostrava,
17. listopadu 15, 708 33, Ostrava-Poruba
tereza.kovarova@vsb.cz

Abstract. A vertex-magic total labeling of a graph $G(V, E)$ is defined as one-to-one mapping from the set of integers $\{1, 2, ..., |V| + |E|\}$ to $V \cup E$ with the property that the sum of the label of a vertex and the labels of all edges adjacent to this vertex is the same constant for all vertices of the graph.
In this article we present some basic ideas of labelings of graphs. We are particularly interested in magic type labelings for cycles. These labelings of cycles allow us to construct vertex magic total labeling of Cartesian products of cycles of any length with odd cycles.

Key words: magic, labeling, vertex-magic total, cycle

1 Introduction

A labeling is a mapping whose domain is some set of graph elements - the set of vertices or the set of all edges and vertices - and whose range is a set of positive integers. There might be various restrictions placed on the mapping. The case which is considered to be usually the most interesting is when the range is the set of consecutive integers starting from 0 or 1 and norepetition is allowed.

There is a wide variety of graph labelings. Some of the most studied are graceful and harmonious labelings, which have a number of applications. Many methods of decompositions of graphs are based on graceful labeling. Harmonious labelings proved to be useful in problems stemming from error correcting codes.

When the idea of magic squares is generalized for graphs, we obtain magic type labelings. The first author who applied ideas of magic square to graphs in early 60's was Sedláček [6]. He defined a graph to be magic if it had an edge-labeling, with range of real numbers such that the sum of the labels around any vertex equals the same constant. Kotzig and Rosa [8] defined magic labeling to be a total labeling if the labels are consecutive integers starting from 1, assigned to both vertices and edges of a graph. For example one can ask whether the sum of labels associated with an edge—label of the edge, and labels on its end-vertices—always add up to the same constant. This case corresponds to edge-magic total labeling. Similarly, to obtain vertex-magic total labeling, sum of the label of a vertex and labels on incident edges must be the same constant independent of the choice of vertex.

Some applications of magic labelings have been studied, mainly in network-related areas, also in the construction of ruler models, which have been applied to the study of radar pulse codes [8].

We will now focus our attention on vertex-magic total labeling.

2 Vertex Magic Total labeling

Our new result, as announced above, is a vertex-magic total (VMT) labeling for an infinite class of graphs - products of cycles. We need to introduce some more notions before we give a method of construction for such VMT labeling.

Definition 1. *Let $G(V, E)$ be a graph with vertex set V and edge set E. A one-to-one mapping $\lambda : V \cup E \to \{1, \ldots, |V| + |E|\}$ is called vertex magic total labeling if there is a constant k such that for every vertex $v \in V$,*

$$\lambda(v) + \sum \lambda(uv) = k$$

where the sum is over all vertices u adjacent to vertex v. The constant k is called the magic constant for λ.

2.1 Vertex Magic Total labeling of cycles of odd length

We give here an example of VMT labeling for odd cycles, which is then used in construction of VMT labeling of products of cycles. A VMT labeling of a cycle is equivalent to an edge magic total (EMT) labeling of a cycle and results on EMT labelings of cycles of any length are due to Kotzig and Rosa [6].

Let C_n have vertices v_i and edges v_iv_{i+1} for $i = 0, 1 \ldots, n-1$. Subscripts will be taken modulo n. For n odd, labels to vertices and edges are assigned as follows:

$$\lambda(v_i) = n - i,$$

$$\lambda(v_iv_{i+1}) = \begin{cases} \frac{i}{2} + 1 & \text{for } i \text{ even,} \\ b + \frac{n+i}{2} + 1 & \text{for } i \text{ odd.} \end{cases}$$

Magic constant is

$$k = \frac{1}{2}(5n + 3).$$

Figure 2.1 shows example of this labeling for C_7.

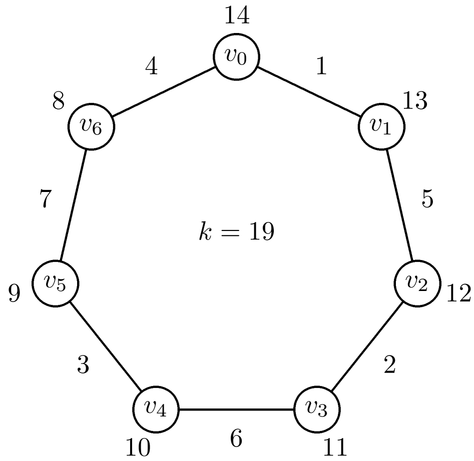

Figure 2.1: *VMT labeling of C_7 with the magic constant 19*

2.2 Vertex-Antimagic Total labeling

In our construction of VMT labeling of products of cycles also antimagic-type labeling appears. We give here definition of this labeling. More about antimagic-type labelings can be found in [1].

Definition 2. *A one-to-one mapping $\lambda : V \cup E \to \{1, \dots, |V|+|E|\}$ is called (a,d)-vertex-antimagic total labeling if the set of sums for every vertex $v \in V$,*

$$\lambda(v) + \sum_{u \in N(v)} \lambda(uv) = k$$

form an arithmetic sequence with the difference d, starting with the smallest sum a.

2.3 VMT labeling of products of cycles of any length with odd cycles

Theorem 1. *For each $m, n \geq 3$ and n odd, there exists a VMT labeling of $C_m \times C_n$ with magic constant*

$$k = \frac{1}{2}m(15n + 1) + 2.$$

Proof. We use an (a,2)-antimagic labeling to assign labels to vertices and edges of vertical cycles (C_m). Labels to edges of horizontal cycles (C_n) are assigned in accordance to the VMT labeling of odd cycles. That means, sums on vertices obtained from the labels of vertical cycles together with labels on horizontal edges follow the same pattern as VMT labeling of odd cycle, which was introduced above.

Let $C_m \times C_n$ have vertices $v_{i,j}$, the vertical edges $v_{i,j}v_{i+1,j}$ and the horizontal edges $v_{i,j}v_{i,j+1}$ where $i = 0, 1, \dots, m-1$ and $j = 0, 1, \dots, n-1$ for $m, n \geq 3$ and n odd. Consider the following labeling, where the subscripts i, j are taken modulo m and n respectively.

$$\begin{aligned}
\lambda(v_{i,j}) &= 3mj + 2 && \text{for} \quad i = 0, \\
&= 3mj + 2m - 2[i-1] && \text{for} \quad i = 1, \dots, m-1\,, \\
\lambda(v_{i,j}v_{i+1,j}) &= 3m\,[n - (j+1)] + 2i + 1, \\
\lambda(v_{i,j}v_{i,j+1}) &= 3m\left(\tfrac{j}{2} + 1\right) - i && \text{for} \quad j \text{ even}, \\
&= 3m\left(\tfrac{n+j}{2} + 1\right) - i && \text{for} \quad j \text{ odd}.
\end{aligned}$$

From the construction it is easy to observe that all $3mn$ numbers are assigned. Using also the labeling for $C_4 \times C_5$ in Figure 2.2, we can easily see that the sequence of labels is divided into m-tuples. If we define the set $a_i = \{m(i-1), (m+1)(i-1), \dots, (2m-1)(i-1)\}$ to be the i-th m-tuple for $i = 1, 2, \dots, n$, the numbers from m-tuples $a_3, a_6, \dots, a_{3n}$ are assigned to the horizontal edges. Numbers from m-tuples $a_1, a_2, a_4, a_5, \dots, a_{3n-2}, a_{3n-1}$ are assigned to the vertices and edges of vertical cycles. Vertices receive even and edges odd labels.

It is easy to verify that the sum of labels at each vertex is the same.

(i) For $i = 0$ and j even we get

$$\begin{aligned}
k &= \lambda(v_{i,j}) + \lambda(v_{i-1,j}v_{i,j}) + \lambda(v_{i,j}v_{i+1,j}) + \lambda(v_{i,j-1}v_{i,j}) + \lambda(v_{i,j}v_{i,j+1}) \\
&= 3mj + 2 + 3m\,[n - (j+1)] + 2(m-1) + 1 + 3m\,[n - (j+1)] + 1 + \\
&\quad 3m\left(\tfrac{n+j-1}{2} + 1\right) + 3m\left(\tfrac{j}{2} + 1\right) \\
&= 3mn + 3mn + \tfrac{3}{2}mn - 3m + 2m - 3m - \tfrac{3}{2}m + 3m + 3m + 2 \\
&= \tfrac{1}{2}m(15n + 1) + 2
\end{aligned}$$

(ii) For $i = 1, \dots, m-1$ and j even the sum of the labels is

$$\begin{aligned}
k &= \lambda(v_{i,j}) + \lambda(v_{i-1,j}v_{i,j}) + \lambda(v_{i,j}v_{i+1,j}) + \lambda(v_{i,j-1}v_{i,j}) + \lambda(v_{i,j}v_{i,j+1}) \\
&= 3mj + 2m - 2[i-1] + 3m\,[n - (j+1)] + 2(i-1) + 1 + 3m\,[n - (j+1)] + 2i + 1 + \\
&\quad 3m\left(\tfrac{n+j-1}{2} + 1\right) - i + 3m\left(\tfrac{j}{2} + 1\right) - i \\
&= 3mn + 3mn + \tfrac{3}{2}mn + 2m - 3m - 3m - \tfrac{3}{2}m + 3m + 3m + 2 \\
&= \tfrac{1}{2}m(15n + 1) + 2
\end{aligned}$$

(iii) and (iv) Similarly we get the same constant also for remaining two cases, $i = 0$ and j odd, or $i = 1, \dots, m-1$ and j odd.

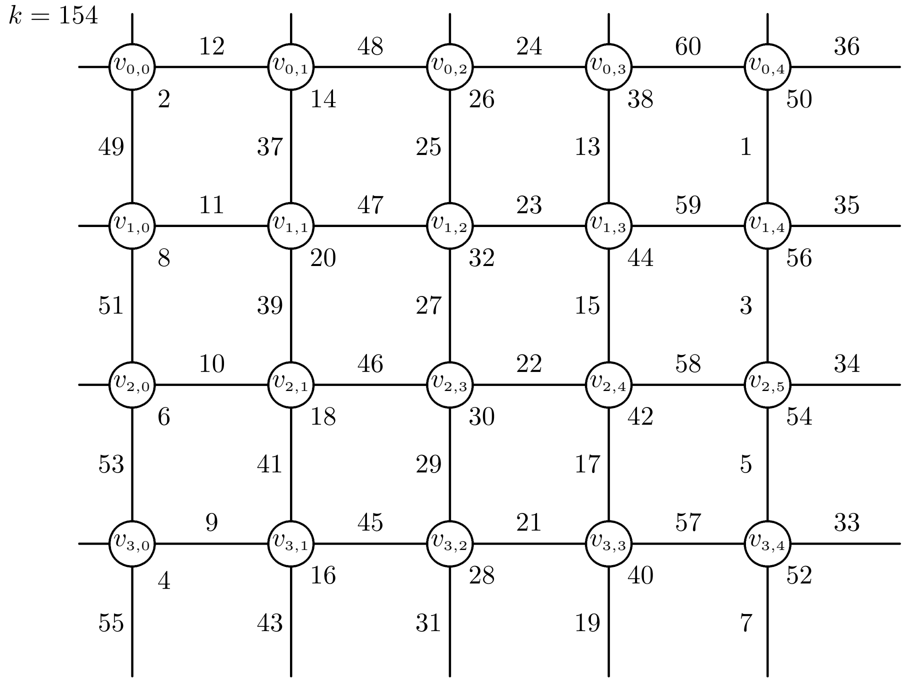

Figure 2.2: *VMT labeling of $C_4 \times C_5$ with the magic constant $k = 154$*

Remark 1. Our labeling can be easily modified to obtain some other values for the magic constant k. The sets of numbers which can be used to label vertices, vertical edges and horizontal edges, so that the same pattern as in the given labeling is followed are shown in Table 1 .

In Table 1 the index t runs from 0 to $n-1$.

Table 1. Intervals from which labels are taken for vertices, vertical and horizontal edges, and corresponding magic constants.

1	$\lambda(v_{i,j})$	$3mt+2,\ 3mt+4, \dots, 3mt+2m$	$k = \frac{1}{2}m(15n+1)+2$
	$\lambda(v_{i,j}v_{i+1,j})$	$3mt+1,\ 3mt+3, \dots, 3mt+2m-1$	
	$\lambda(v_{i,j}v_{i,j+1})$	$3mt+2m+1,\ 3mt+2m+2, \dots, 3mt+3m$	
2	$\lambda(v_{i,j})$	$3mt+1,\ 3mt+3, \dots, 3mt+2m-1$	$k = \frac{1}{2}m(15n+1)+3$
	$\lambda(v_{i,j}v_{i+1,j})$	$3mt+2,\ 3mt+4, \dots, 3mt+2m$	
	$\lambda(v_{i,j}v_{i,j+1})$	$3mt+2m+1,\ 3mt+2m+2, \dots, 3mt+3m$	
3	$\lambda(v_{i,j})$	$3mt+m+2,\ 3mt+m+4, \dots, 3mt+3m$	$k = \frac{1}{2}m(15n-1)+2$
	$\lambda(v_{i,j}v_{i+1,j})$	$3mt+m+1,\ 3mt+m+3, \dots, 3mt+3m-1$	
	$\lambda(v_{i,j}v_{i,j+1})$	$3mt+1,\ 3mt+2, \dots, 3mt+m$	
4	$\lambda(v_{i,j})$	$3mt+m+1,\ 3mt+m+3, \dots, 3mt+3m-1$	$k = \frac{1}{2}m(15n-1)+3$
	$\lambda(v_{i,j}v_{i+1,j})$	$3mt+m+2,\ 3mt+m+4, \dots, 3mt+3m$	
	$\lambda(v_{i,j}v_{i,j+1})$	$3mt+1,\ 3mt+2, \dots, 3mt+m$	

Remark 2. We cannot easily extend this method for VMT labeling of $C_m \times C_n$ with n even. The reason is that there is no known ïiceV̈MT labeling of even cycles. Nice labeling in the sense that labels on vertices or on edges are consecutive integers or follow some regular pattern.

3 Conclusion

This result on VMT labelings of products of cycles was published in proceedings of conference MIGHTY XXXVI [2], together with another method of construction of VMT labeling for products of cycles of odd length. The other method allows to extend spectrum of magic constants, which can be realized for products of cycles of odd length. The ideas of given methods proved to be generalizable also for constructions of VMT labelings of other infinite classes of graphs. Some of these results have been already presented [3], [5].

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