Decidability of Bisimulation Equivalence for First-Order Grammars

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July 1, 2011

Abstract

A self-contained proof of the decidability of bisimulation equivalence for first-order grammars is given. This provides an alternative for Sénizergues' decidability proof (1998,2005) for nondeterministic pushdown automata with deterministic popping ε-steps, which generalized his decidability proof for language equivalence of deterministic pushdown automata.

1 Introduction

Remark. This is a *preliminary version* which aims to provide a complete proof, while giving only a (very) partial list of references to related work.

Ginsburg and Greibach posed the decidability question for language equivalence of deterministic pushdown automata (dpda) in 1960s. Later the decidability question for a more general problem, namely for bisimulation equivalence of processes generated by nondeterministic pushdown automata, was posed (in 1990s, probably first formulated by Caucal). Both questions were solved positively by Sénizergues [2, 3].

In (the first versions of) [1], both problems were considered in the framework of first-order grammars, with the aim to provide shorter alternatives for Sénizergues' proofs. The first part of [1] showed the decidability of trace equivalence for deterministic first-order grammars (which yields the decidability of language equivalence for dpda). Sénizergues [4] demonstrated a mistake in the second part (of [1], version 3, Dec 2010) which aimed to generalize the first part to bisimulation equivalence for general first-order grammars.

The aim of this paper is to repair this mistake, by giving a self-contained proof of the decidability of bisimulation equivalence for first-order grammars. The proof for trace equivalence of deterministic first-order grammars can be seen as a special case of the (more general) proof presented here.

The structure of the paper is clear from the (sub)section titles.

2 Basic definitions and useful observations

By $\mathbb{N}$ we denote the set $\{0, 1, 2, \ldots\}$ of natural numbers. Symbol ω is understood as an infinite number satisfying $n < \omega$ and $\omega - n = \omega + n = \omega$ for any $n \in \mathbb{N}$.
For a set $\mathcal{A}$ and $\ell \in \mathbb{N}$, by $\mathcal{A}^{\leq \ell}$ we denote the set of sequences (words) $w = a_1 a_2 \ldots a_m$ where $a_i \in \mathcal{A}$ and $m \leq \ell$. We thus have $\mathcal{A}^{\leq \ell} = \{w \in \mathcal{A}^* \mid |w| \leq \ell\}$ where $|w|$ denotes the length of w and $\mathcal{A}^*$ the set of finite sequences of elements of $\mathcal{A}$. By ε we denote the empty sequence, hence $|\varepsilon| = 0$. $\mathcal{A}^\omega$ is the set of infinite sequences of elements of $\mathcal{A}$ (i.e., the set of mappings $\mathbb{N} \to \mathcal{A}$). $L \subseteq \mathcal{A}^*$ is *prefix closed* if $uv \in L$ implies $u \in L$. A *word* $w \in L$ is *maximal* (in L) if there is no $v \neq \varepsilon$ such that $wv \in L$. By $u \backslash L$ we denote the *left quotient* of L by u, i.e. the set $u \backslash L = \{v \in \mathcal{A}^* \mid uv \in L\}$.
Remark. In the deterministic case, DetCase for short, some notions and arguments are a bit simpler. The following text contains some hints which should help when the reader wishes to concentrate just on DetCase.

Labelled transition systems and (stratified) bisimulation equivalence

A *labelled transition system (LTS)* is a tuple $\mathcal{L} = (\mathcal{S}, \mathcal{A}, (\xrightarrow{a})_{a \in \mathcal{A}})$ where $\mathcal{S}$ is the set of *states*, $\mathcal{A}$ the set of *actions* and $\xrightarrow{a} \subseteq \mathcal{S} \times \mathcal{S}$ is the set of *transitions labelled with a*. $\mathcal{L} = (\mathcal{S}, \mathcal{A}, (\xrightarrow{a})_{a \in \mathcal{A}})$ is *image-finite* if the set $\{s' \mid s \xrightarrow{a} s'\}$ is finite for every pair $s \in \mathcal{S}$, $a \in \mathcal{A}$; moreover, if $\{s' \mid s \xrightarrow{a} s'\}$ has at most one element (for every pair s, a) then $\mathcal{L}$ is *deterministic*, a *det-LTS* for short.

The relations $\xrightarrow{w} \subseteq \mathcal{S} \times \mathcal{S}$ for $w \in \mathcal{A}^*$ are defined inductively: $s \xrightarrow{\varepsilon} s$; if $s \xrightarrow{a} s_1$ and $s_1 \xrightarrow{v} s_2$ then $s \xrightarrow{av} s_2$. We say that s' is *reachable from s by a word w* if $s \xrightarrow{w} s'$. A *state* $s \in \mathcal{S}$ *enables* (a trace) $w \in \mathcal{A}^*$, denoted $s \xrightarrow{w}$, if $s \xrightarrow{w} s'$ for some s'.

(In DetCase we can now jump to Proposition 1 and take it as a definition.)
A relation $R \subseteq \mathcal{S} \times \mathcal{S}$ is a *bisimulation* if R is symmetric ($(s, t) \in R \Rightarrow (t, s) \in R$) and for every $(s, t) \in R$ we have: if $s \xrightarrow{a} s'$ then there is t' such that $t \xrightarrow{a} t'$ and $(s', t') \in R$. Two *states* s, t are *bisimilar*, written $s \sim t$, if there is a bisimulation R containing (s, t); hence $\sim$ is the union of all bisimulations on $\mathcal{S}$. Relations $\sim_0$, $\sim_1$, $\sim_2$, ..., are defined as follows: $\sim_0 = \mathcal{S} \times \mathcal{S}$; $\sim_{k+1}$ is the maximal symmetric relation satisfying: if $s \sim_{k+1} t$ and $s \xrightarrow{a} s'$ then there is t' such that $t \xrightarrow{a} t'$ and $s' \sim_k t'$. We recall some standard facts.

Proposition 1 *If $\mathcal{L}$ is a det-LTS then $s \sim_k t$ iff $\forall w \in \mathcal{A}^{\leq k} : s \xrightarrow{w} \Leftrightarrow t \xrightarrow{w}$, and $\sim = \bigcap_{k \in \mathbb{N}} \sim_k$.*

Proposition 2 *(1) Relations $\sim_k$ ($k \in \mathbb{N}$) and $\sim$ are equivalences.*
(2) $\sim_0 \supseteq \sim_1 \supseteq \sim_2 \supseteq \cdots$. (3) $\cap_{k \in \mathbb{N}} \sim_k \supseteq \sim$. (4) If $\mathcal{L}$ is image-finite then $\cap_{k \in \mathbb{N}} \sim_k = \sim$.

In image-finite systems we define the *equivalence-level* for each pair of states as follows:

$$\text{EqLv}(s, t) = k \ (k \in \mathbb{N}) \text{ if } s \sim_k t \text{ and } s \not\sim_{k+1} t\,; \ \ \text{EqLv}(s, t) = \omega \text{ if } s \sim t.$$

Observation 3 *If* $\text{EqLv}(s, t) = k$ *and* $\text{EqLv}(s, s') \geq k + 1$ *then* $\text{EqLv}(s', t) = k$.

Regular terms and their finite graph presentations

In the LTSs which we will consider, the states are regular (first-order) terms, i.e. finite or infinite terms with only finitely many subterms (where a subterm can have infinitely many occurrences). We assume a (globally) fixed countable set $\mathcal{V} = \{x_1, x_2, x_3, \ldots\}$ of (first-order) *variables*. The terms are defined over a given *finite* set $\mathcal{N}$ of ranked symbols, called *nonterminals* (which can be viewed as function symbols); each $Y \in \mathcal{N}$ has $arity(Y) \in \mathbb{N}$.

We now recall a variant of the standard definitions. A *regular term* over $\mathcal{N}$, a *term* for short, can be viewed as a partial mapping $E : \mathbb{N}^* \to \mathcal{N} \cup \mathcal{V}$ (where $\text{DOM}(E)$ is nonempty and prefix-closed) which can be presented by the following finite-graph presentation.
A *graph presentation* (of a term) is a structure $\text{GP} = (\text{NODES}, \lambda, \text{SUCC}, \text{ROOT})$ where NODES is a finite set of *nodes*, $\lambda : \text{NODES} \to \mathcal{N} \cup \mathcal{V}$, $\text{SUCC}(\mathtt{n}, i) \in \text{NODES}$ for each $\mathtt{n} \in \text{NODES}$ and $1 \leq i \leq arity(\lambda(\mathtt{n}))$, and $\text{ROOT} \in \text{NODES}$ is the *root*; for $x_i \in \mathcal{V}$ we put $arity(x_i) = 0$. A pair $(\mathtt{n}, \text{SUCC}(\mathtt{n}, i))$ is called a (directed) *edge* (from $\mathtt{n}$ to $\text{SUCC}(\mathtt{n}, i)$) labelled with i. A (finite) *path from* $\mathtt{n}_0$ is a sequence $\mathtt{n}_0, i_1, \mathtt{n}_1, i_2, \mathtt{n}_2, \ldots, i_d, \mathtt{n}_d$ $(d \geq 0)$ where $\text{SUCC}(\mathtt{n}_{j-1}, i_j) = \mathtt{n}_j$ for $j = 1, 2, \ldots, d$; we call $i_1 i_2 \ldots i_d$ the *edge-label sequence* of this path and n_d its *end-node*. GP represents the term E_{GP} where $\text{DOM}(E_{\text{GP}})$ consists of the edge-label sequences of the (finite) paths from ROOT; $E_{\text{GP}}\langle\gamma\rangle = \lambda(\mathtt{n})$ where $\mathtt{n}$ is the end-node of the path with the edge-label sequence γ. (It is useful to draw some illustrating figures, and identify *terms* with possibly infinite *trees*.)

Observation 4 *Given graphs* GP_1, GP_2, *we can effectively decide if* $E_{\text{GP}_1} = E_{\text{GP}_2}$.

Some other effective operations with graph presentations will be left implicit. We also leave implicit that our further notions (like a d-depth subterm, the result of substitution, H^{lim_i} in Definition 5, etc.) are independent of particular presentations of terms.

Assume GP and a path from the root to an end-node $\mathtt{n}$, where γ is the edge-label sequence and $|\gamma| = d$ $(d \geq 0)$. By taking $\mathtt{n}$ as the root, we get a graph presentation GP' of a *d-depth subterm* of E_{GP}; one *occurrence of* this *subterm* is thus *at* γ. We note that the number of d-depth subterms of E is bounded by c^d where $c = \max\{ arity(Y) \mid Y \in \mathcal{N} \}$.

A *term* E is *finite* if $\text{DOM}(E)$ is finite (so if the paths from the root in graph presentations of E are acyclic). For a finite term E we put $\text{DEPTH}(E) = 1 + \max\{ |\gamma| \mid \gamma \in \text{DOM}(E) \}$. We note that if F is a d-depth subterm of a finite E then $\text{DEPTH}(F) \leq \text{DEPTH}(E) - d$.

We define the *size of a graph presentation* GP as the number of nodes of GP, or as the largest index i of variables x_i labelling the nodes in GP if this is bigger. Hence there are only finitely many graph presentations with the size bounded by a given $s \in \mathbb{N}$. $\text{PRESSIZE}(F)$ denotes the *presentation size* of a term F, i.e. the size of the least graph presentation of F. We put $\text{PRESSIZE}(E, F) = \text{PRESSIZE}(E) + \text{PRESSIZE}(F)$ for pairs of terms.

Convention. For terms we usually use symbols E, F, G, H (with subscripts etc.) but also T, U, V, W. It might be sometimes helpful to think about T, U, V, W as of ground terms (with no variables) but we do not stipulate this. Y is used for nonterminals.
By $\text{TERMS}_{\mathcal{N}}$ we denote the set of all terms (i.e. of all regular terms) over $\mathcal{N}$.

Substitutions and a limit substitution

We also use the standard presentation of terms, like $E = Yx_1 \dots x_m$, $E = YF_1 \dots F_m$ where $Y \in \mathcal{N}$, $m = arity(Y)$; also each variable x_i is viewed as a term. By $root(E)$ we denote the *root-nonterminal* Y in the first case, and x_i in the second.

By writing $E[F_1/x_{i_1}, \dots, F_n/x_{i_n}]$ (where $i_j \neq i_{j'}$ if $j \neq j'$) we mean the term, called also an *instance* of E, which arises from E by applying the denoted *substitution*:
for graph presentations GP, GP_1, ..., GP_n of terms E, $F_1, \dots, F_n$, respectively, a graph presentation GP′ of $E[F_1/x_{i_1}, \dots, F_n/x_{i_n}]$ arises by taking the disjoint union of GP, GP_1, ..., GP_n and redirecting each edge leading to (a node labelled with) x_{i_j} in GP to the root of GP_j; the root of GP′ is the root of GP unless $E = x_{i_j}$ in which case it is the root of GP_j.
Convention. When writing $E(F_1, \dots, F_n)$ we mean $E[F_1/x_{i_1}, \dots, F_n/x_{i_n}]$ for an implicit tuple $(x_{i_1}, \dots, x_{i_n})$; we always assume $(x_{i_1}, \dots, x_{i_n}) = (x_1, \dots, x_n)$ if not said otherwise.
(We use $E\langle\gamma\rangle$ for $\gamma \in \text{DOM}(E)$ to avoid a possible confusion with $E(F)$.)

We will need the limit for a particular repeatedly applied substitution:

Definition 5 *Given (a regular term) H we define $H[H/x_i]^\omega$, denoted as H^{lim_i}, as follows: given a graph presentation* GP *where $E_{\text{GP}} = H$, then $H^{lim_i} = E_{\text{GP}'}$ where* GP′ *arises from* GP *by redirecting each edge leading to x_i so that it leads to the root. Hence if $H = x_i$ or if x_i does not occur in H then $H^{lim_i} = H$.*

Observation 6 $H^{lim_i} = H[H^{lim_i}/x_i]$.
$E(V_1, \dots, V_{n-1}, H^{lim_n}(V_1, \dots, V_{n-1})) = E'(V_1, \dots, V_{n-1})$ *where* $E' = E[H^{lim_n}/x_n]$.

First-order grammars as generators of LTSs

Definition 7 *A* first-order grammar *is a tuple $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ where $\mathcal{N}$ is a finite set of ranked* nonterminals, *$\mathcal{A}$ is a finite set of* actions *(or terminals), and $\mathcal{R}$ a finite set of* (root rewriting) rules *r of the form*

$$r : Yx_1x_2 \dots x_m \xrightarrow{a} E \tag{1}$$

where $Y \in \mathcal{N}$, $arity(Y) = m$, $a \in \mathcal{A}$, and E is a finite *term over $\mathcal{N}$ in which each occurring variable is from the set $\{x_1, x_2, \dots, x_m\}$. ($E = x_i$, where $1 \leq i \leq m$, is an example.)*
We put $\text{ACT}(r) = a$, thus defining the mapping $\text{ACT} : \mathcal{R} \to \mathcal{A}$.
$\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ is deterministic *if there is at most one rule (1) for each pair $Y \in \mathcal{N}$, $a \in \mathcal{A}$.*

Remark. Context-free grammars in Greibach normal form can be seen as a special case, where each nonterminal has arity 1. Classical rules like $A \to aBC$, $B \to b$ can be presented as $Ax_1 \xrightarrow{a} BCx_1$, $Bx_1 \xrightarrow{b} x_1$. We only use left derivations, as clarified below.
It might be elegant to introduce also the *empty term* but this is not necessary for our aims.
In DetCase, we only consider deterministic grammars.

Definition 8 *$\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ generates (the rule based) $LTS^{\text{R}}_{\mathcal{G}} = (\text{Terms}_{\mathcal{N}}, \mathcal{R}, (\xrightarrow{r})_{r \in \mathcal{R}})$: for each rule $r : Yx_1x_2 \dots x_m \xrightarrow{a} E$ (recall (1)) we have*

$F \xrightarrow{r} H$ *if* $F = YG_1 \dots G_m$ *and* $H = E(G_1, \dots, G_m)$ *for a tuple of terms* $G_1, \dots, G_m$.
(Note that $(G_1, \dots, G_m) = (x_1, \dots, x_m)$ *yields* $Yx_1 \dots x_m \xrightarrow{r} E$.*)*
For (the action-based) $LTS^{\mathrm{A}}_{\mathcal{G}} = (\mathrm{TERMS}_{\mathcal{N}}, \mathcal{A}', (\xrightarrow{a})_{a \in \mathcal{A}'})$ *we define* $\mathcal{A}' = \mathcal{A} \cup \{a_{x_i} \mid x_i \in \mathcal{V}\}$ *where* a_{x_i} *is a unique (fresh) action attached to* x_i. *For* $a \in \mathcal{A}'$ *we have* $F \xrightarrow{a} H$ *if* $F \xrightarrow{r} H$ *for some* $r \in \mathcal{R}$ *with* $\mathrm{ACT}(r) = a$ *or if* $F = H = x_i$ *and* $a = a_{x_i}$.

Observation 9 $LTS^{\mathrm{R}}_{\mathcal{G}}$ *is a det-LTS for any* $\mathcal{G}$*; variables* x_i *are here (examples of)* dead terms *(not enabling any transition).* $LTS^{\mathrm{A}}_{\mathcal{G}}$ *is a det-LTS iff* $\mathcal{G}$ *is deterministic.*

We have introduced the transitions $x_i \xrightarrow{a_{x_i}} x_i$ in $LTS^{\mathrm{A}}_{\mathcal{G}}$ for later technical convenience (in particular for Point 1. in Proposition 27). We now concentrate on (the rule-based) $LTS^{\mathrm{R}}_{\mathcal{G}}$.

Convention. By u, v, w we denote elements of $\mathcal{R}^*$, or of $\mathcal{R}^\omega$, if not said otherwise.
Given GP presenting $F = YU_1 \dots U_m$ and a rule r in the form (1), for creating GP$'$ presenting H, where $F \xrightarrow{r} H$, it is sufficient to extend GP with GP$''$ presenting (the finite) E where we redirect the edges leading to x_i to the appropriate (root-successor) nodes in GP. The root of GP$''$ is the root of GP$'$ unless $E = x_i$ in which case the root in GP is shifted along the edge i. We can thus vividly observe that if $F \xrightarrow{w} H$ then only variables occurring in F might (but might not) occur in H; also note that if F is finite then H is finite.

Observation 10 *(1.) If* $F \xrightarrow{w} H$ *then* $F(G_1, \dots, G_n) \xrightarrow{w} H(G_1, \dots, G_n)$.
(2.) Given $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$*, we can compute constants* $\mathrm{SIZEINC}$, $\mathrm{DINC} \geq 1$ *which satisfy: if* $E \xrightarrow{w} F$ *then* $\mathrm{PRESSIZE}(F) \leq \mathrm{PRESSIZE}(E) + |w| \cdot \mathrm{SIZEINC}$*; moreover, if* E *is finite then (*F *is finite and)* $\mathrm{DEPTH}(F) \leq \mathrm{DEPTH}(E) + |w| \cdot \mathrm{DINC}$.
(3.) If $F \xrightarrow{w} F'$ *where* $|w| \leq d$ *then* $F = H(V_1, \dots, V_n)$, $F' = H'(V_1, \dots, V_n)$ *where* H *is finite,* $\mathrm{DEPTH}(H) \leq 1 + d$, $H \xrightarrow{w} H'$ *(hence* $\mathrm{DEPTH}(H') \leq 1 + d + d \cdot \mathrm{DINC}$*), and each* V_i *is either a* d*-depth subterm of* F *or a variable occurring in* F *in depth less than* d.

Sink words and some useful technical propositions

Assuming a given $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$, we now introduce some useful notions and note their properties. The main aim is to show Proposition 15 which is a bit technical (and might be skipped at first reading), nevertheless thinking over the details might help to better understand (the deterministic) $LTS^{\mathrm{R}}_{\mathcal{G}}$ as well as the later proof.

By referring to a *segment* $E \xrightarrow{w}$, where $w = r_1 r_2 \dots r_k \in \mathcal{R}^*$ or $w = r_1 r_2 \dots \in \mathcal{R}^\omega$, we mean that w is enabled in E and we refer to the unique sequence $E \xrightarrow{r_1} E_1 \xrightarrow{r_2} E_2 \xrightarrow{r_3} \cdots$.

We call $w \in \mathcal{R}^*$ a (Y, j)*-sink-word*, where $1 \leq j \leq arity(Y) = m$, if $Yx_1 \dots x_m \xrightarrow{w} x_j$.

Observation 11 *If* w *is a* (Y, j)*-sink-word then* $w = rv$*, for* $r : Yx_1 \dots x_m \xrightarrow{a} E$*, where*
(1.) either $E = x_j$ *and* $v = \varepsilon$ *(thus* $|w| = 1$ *since* $w = r$*),*
(2.) or $E = Y_0 F_1 \dots F_{m_0}$ *and there is* $\gamma = i_1 i_2 \dots i_k \in \mathrm{DOM}(E)$ *(recall* $E : \mathbb{N}^* \to \mathcal{N} \cup \mathcal{V}$*) where* $E\langle\gamma\rangle = x_j$*,* $E\langle i_1 \dots i_\ell\rangle = Y_\ell$ *for* $\ell = 1, 2 \dots, k{-}1$*, and* $w = rv = rw_1 w_2 \dots w_k$ *where* w_ℓ *is an* $(Y_{\ell-1}, i_\ell)$*-sink word for* $\ell = 1, 2 \dots, k$ *(informally:* $E \xrightarrow{w_1 \dots w_k}$ *is sinking along* $Y_0{-}i_1{-}Y_1{-}i_2{-}Y_2{-}\cdots{-}Y_{k-1}{-}i_k{-}x_j$*).*

A *segment* $E \xrightarrow{w}$ is a *sink* if $E = YU_1 \dots U_m$ and w is a (Y,j)-sink-word for some j (in which case we have $E \xrightarrow{w} U_j$). $E \xrightarrow{w}$ *does not sink* if there is no sink $E \xrightarrow{v}$ where v is a prefix of w ($w \in \mathcal{R}^* \cup \mathcal{R}^\omega$); hence if $w \neq \varepsilon$ then $root(E) = Y \in \mathcal{N}$ and $Yx_1 \dots x_m \xrightarrow{w}$.

Observation 12 *If* $E \xrightarrow{u} E'$ *is a sink and* $E \xrightarrow{u'} V \xrightarrow{u''} E'$ *where* $u'u'' = u$, $u'' \neq \varepsilon$, *then there is a prefix* v *of* u'' *such that* $V \xrightarrow{v}$ *is a sink.*

Later we look at the shortest (Y,j)-sink words. We could compose them to get the shortest $(Y_0, i_1, Y_1, i_2, Y_2, \dots, Y_{k-1}, i_k)$-sink words, but more crude notions suffice for our aims.
For $w \in \mathcal{R}^*$, a *segment* $E_0 \xrightarrow{w}$ is ℓ*-sinking* ($\ell \in \mathbb{N}$) if $w = u_1u_2 \dots u_k v$ where

$$E_0 \xrightarrow{u_1} E_1 \xrightarrow{u_2} E_2 \xrightarrow{u_3} \cdots E_{k-1} \xrightarrow{u_k} E_k \xrightarrow{v} E_0' \,, \tag{2}$$

$|u_i| \leq \ell$ and $E_{i-1} \xrightarrow{u_i} E_i$ is a sink for $1 \leq i \leq k$, and $|v| < \ell$ and $E_k \xrightarrow{v}$ does not sink.

Proposition 13 *In the sequence* (2)*:*
(1.) Each E_i, $0 \leq i \leq k$, *is an* i*-depth subterm of* E_0.
(2.) If $|w| = \ell_1$ *then some* E_i *is a* $(\ell_1\, div\, \ell)$*-depth subterm of* E_0.
(3.) For any V *such that* $E_0 \xrightarrow{w_1} V \xrightarrow{w_2} E_0'$ *where* $w_1w_2 = w$ *we have* $E_0' = G(V_1, \dots, V_m)$ *for some subterms* $V_1, \dots, V_m$ *of* V *and some finite* G *with* $\text{DEPTH}(G) \leq 2 + (\ell{-}1) \cdot \text{DINC}$.

Proof: (1.), (2.) are obvious. We assume $E_0 \xrightarrow{w_1} V \xrightarrow{w_2} E_0'$ and show (3.). If E_0' is a subterm of V, we are done (G is just a variable). Otherwise we must have $E_0 \xrightarrow{v_1} V' \xrightarrow{v_2} E_0'$ where $v_1v_2 = w$, V' is a subterm of V and $V' \xrightarrow{v_2}$ does not sink; by Observation 12, v_2 is a suffix of v and thus $|v_2| \leq (\ell{-}1)$. Hence $V' = YV_1 \dots V_m$ and $Yx_1 \dots x_m \xrightarrow{v_2} G$ where $\text{DEPTH}(G) \leq 2 + (\ell{-}1) \cdot \text{DINC}$ (and $V' = YV_1 \dots V_m \xrightarrow{v_2} G(V_1, \dots, V_m) = E_0'$). □

Consider a presentation $E_0' = G(V_1, \dots, V_m)$ arising by applying Point (3.) for $V = E_0$ (so $V_1, \dots, V_m$ are subterms of E_0). We observe that if the (above) ℓ-sinking segment $E_0 \xrightarrow{w} E_0'$ is followed by another ℓ-sinking segment $E_0' \xrightarrow{w'} E_0''$ with $|w| = |w'| = \ell_1$ for ℓ_1 sufficiently bigger than ℓ then this second segment 'erases' the possible 'head' G in $E_0' = G(V_1, \dots, V_m)$ (by "sinking along a path in $\text{DOM}(G)$"), while reaching a subterm V_j of E_0, and E_0'' is thus also presentable as $E_0'' = G'(V_1', \dots, V_{m'}')$ where $\text{DEPTH}(G') \leq 2 + (\ell{-}1) \cdot \text{DINC}$ and $V_1', \dots, V_{m'}'$ are subterms of E_0. In the next proposition we put the lower bound for ℓ_1 so that erasing also slightly bigger heads is guaranteed.

Proposition 14 *Suppose some* ℓ_0, ℓ_1 *where* $1 \leq \ell_0 \cdot (3 + 2\ell_0 \cdot \text{DINC}) \leq \ell_1$.
(1.) If $E \xrightarrow{w} E'$ *is* ℓ_0*-sinking, where* $|w| = \ell_1$, *then there is a* d*-depth subterm* F *of* E *where* $d > 2 + 2\ell_0 \cdot \text{DINC}$ *and* $E \xrightarrow{w'} F \xrightarrow{w''} E'$ *for some* w', w'', *where* $w = w'w''$.
Moreover, if $E = H(U_1, \dots, U_n)$ *for a finite* H *then either* $H \xrightarrow{u} x_j$ *(*U_j *is exposed) for some prefix* u *of* w *(which surely happens when* $\text{DEPTH}(H) \leq 2 + 2\ell_0 \cdot \text{DINC}$*) or* $H \xrightarrow{w} H'$ *where* $\text{DEPTH}(H') < \text{DEPTH}(H)$ *(and* $E' = H'(U_1, \dots, U_n)$*).*
(2.) If $T_0 \xrightarrow{w_1} T_1 \xrightarrow{w_2} T_2 \xrightarrow{w_3} \cdots$ *where* $|w_i| = \ell_1$ *and* $T_{i-1} \xrightarrow{w_i} T_i$ *is* ℓ_0*-sinking for all* $i = 1, 2, 3, \dots$ *then the set* $\{T_i \mid i \in \mathbb{N}\}$ *is finite.*

Proof: (1.) follows from Proposition 13(2.) and from noting that $\text{DEPTH}(H') \leq \text{DEPTH}(H) - (3 + 2\ell_0 \cdot \text{DINC}) + (\ell_0{-}1) \cdot \text{DINC} < \text{DEPTH}(H)$.
(2.) We have, in fact, already observed that each T_i can be presented as $G(V_1, \ldots, V_m)$ where $\text{DEPTH}(G) \leq 2 + (\ell_0{-}1) \cdot \text{DINC}$ and $V_1, \ldots, V_m$ are subterms of T_0. □

The next proposition also uses B_i for some terms (later called balancing pivots).

Proposition 15 *Let us have ℓ_0, ℓ_1 where $1 \leq \ell_0 \cdot (3 + 2\ell_0 \cdot \text{DINC}) \leq \ell_1$ (as previously). Assume an infinite sequence $B_1 \xrightarrow{w_1} B_2 \xrightarrow{w_2} B_3 \xrightarrow{w_3} \cdots$ where each segment $B_i \xrightarrow{w_i} B_{i+1}$ can be presented as $B_i \xrightarrow{u} W_0 \xrightarrow{v_1} W_1 \xrightarrow{v_2} \cdots \xrightarrow{v_{k-1}} W_{k-1} \xrightarrow{v_k} B_{i+1}$ where $|u| \leq 2\ell_1$, $|v_1| = |v_2| = \cdots = |v_k| = \ell_1$ and each $W_{j-1} \xrightarrow{v_j} W_j$ is ℓ_0-sinking, for $j = 1, 2, \ldots, k$ where we put $W_k = B_{i+1}$. Then*

1. *either $B_{i_0} = B_{i_1} = B_{i_2} = \cdots$ for some infinite sequence $1 \leq i_0 < i_1 < i_2 < \cdots$,*
2. *or there are u, v_1, v_2 and a ('stair-base') term V such that for some $k \geq 1$ we have $B_1 \xrightarrow{u} B_k \xrightarrow{v_1} V \xrightarrow{v_2} B_{k+1} \xrightarrow{w_{k+1}} B_{k+2} \xrightarrow{w_{k+2}} B_{k+3} \xrightarrow{w_{k+3}} \cdots$ where $u = w_1 w_2 \ldots w_{k-1}$, $v_1 v_2 = w_k$, and $V \xrightarrow{w'}$ does not sink, for $w' = v_2 w_{k+1} w_{k+2} \ldots$;*
   hence $V = Y U_1 \ldots U_m$ where $Y x_1 \ldots x_m \xrightarrow{v_2} H_0 \xrightarrow{w_{k+1}} H_1 \xrightarrow{w_{k+2}} H_2 \xrightarrow{w_{k+3}} \cdots$ and $B_{k+1} = H_0(U_1, \ldots, U_m)$, $B_{k+2} = H_1(U_1, \ldots, U_m)$, $B_{k+3} = H_2(U_1, \ldots, U_m)$, ... where H_j are finite terms with $\text{DEPTH}(H_j) \leq g_{\ell_1}(j) = 2 + (1 + j) \cdot 2\ell_1 \cdot \text{DINC}$.

Proof: If $B_i \xrightarrow{u_1} V \xrightarrow{u_2} B_{i+1}$ where $u_1 u_2 = w_i$ and V is a subterm of B_1 then $B_{i+1} = G(V_1, \ldots, V_m)$ where $V_1, \ldots, V_m$ are subterms of B_1 and $\text{DEPTH}(G) \leq 2 + 2\ell_1 \cdot \text{DINC}$ (which easily follows from Propositions 10(2), 14(1) and 13(3)). So if some subterm V of B_1 occurs infinitely often in the (infinite) segment $B_1 \xrightarrow{w}$, where $w = w_1 w_2 w_3 \ldots$, then infinitely many B_i are the same. Otherwise there is the longest prefix u' of w such that $B_1 \xrightarrow{u'} V$ where V is a subterm of B_1. Denoting $w = u'w'$ we get that $V \xrightarrow{w'}$ does not sink (otherwise u was not the longest). The rest can be easily checked (by Propositions 10(2) and 14(1)). □

Number M_0 bigger than the length of the shortest (Y, j)-sink-words

Convention. (Given $\mathcal{G}$,) for each Y, j ($1 \leq j \leq arity(Y)$) we fix $\text{SSW}(Y, j)$ which is a shortest (Y, j)-sink word; we put $\text{SSW}(Y, j) = \varepsilon$ if no (Y, j)-sink word exists. We also put

$$M_0 = 1 + \max\{\, |\text{SSW}(Y, j)| \mid Y \in \mathcal{N}, 1 \leq j \leq arity(Y)\,\}. \tag{3}$$

The next proposition shows that the above task can be done effectively. It can be proved by using standard (dynamic programming) techniques, when recalling Observation 11. (Also a brute force search, with a termination argument based on Observation 11, would do.)

Proposition 16 *There is an algorithm which, given $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$, computes the shortest (Y, j)-sink words (or finds that there are none) for each pair Y, j.*

D-strategies; semidecidability of $T \overset{?}{\not\sim} U$

Given $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$, by $\sim$ and $\sim_k$ ($k \in \mathbb{N}$) we refer to the equivalences on $\textsc{Terms}_{\mathcal{N}}$ induced by the (action-based) labelled transition system $LTS^{\mathrm{A}}_{\mathcal{G}} = (\textsc{Terms}_{\mathcal{N}}, \mathcal{A}', (\xrightarrow{a})_{a \in \mathcal{A}'})$; this also yields $\textsc{EqLv}(E, F)$ for pairs of terms. In particular note that $x_i \not\sim_1 E$ if $E \neq x_i$ (due to $x_i \xrightarrow{a_{x_i}} x_i$).

Our decision problem asks, given $\mathcal{G}$ and an input pair (T_{in}, U_{in}) (of regular terms) if $T_{in} \sim U_{in}$. We will first note that the complementary problem, asking if $T_{in} \not\sim U_{in}$, i.e. if $\exists k : T_{in} \not\sim_k U_{in}$, is semidecidable. We start with a piece of notation.

Definition 17 *Given $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$, we define the set $\textsc{er}(E) \subseteq \mathcal{R}$ of* enabled rules for term E *as $\textsc{er}(E) = \{r \in \mathcal{R} \mid E \xrightarrow{r}\}$ (which is determined by $root(E)$, being $\emptyset$ if $E = x_i$).*

We also define $\textsc{round} = \{\, (r, r') \mid r, r' \in \mathcal{R}, \textsc{act}(r) = \textsc{act}(r')\}$.

For $\pi = (r, r')$ we put $\pi_{\mathrm{L}} = r$, $\pi_{\mathrm{R}} = r'$; for a sequence $\alpha \in \textsc{round}^*$ we define $\alpha_{\mathrm{L}}, \alpha_{\mathrm{R}} \in \mathcal{R}^*$ inductively: $\varepsilon_{\mathrm{L}} = \varepsilon_{\mathrm{R}} = \varepsilon$, $(\pi\beta)_{\mathrm{L}} = \pi_{\mathrm{L}}\beta_{\mathrm{L}}$, $(\pi\beta)_{\mathrm{R}} = \pi_{\mathrm{R}}\beta_{\mathrm{R}}$.
We further let $\alpha, \beta, \gamma, \ldots$ range over $\textsc{round}^*$.
We use notation $(T, U) \xrightarrow{\alpha} (T', U')$ as a shorthand meaning $T \xrightarrow{\alpha_{\mathrm{L}}} T'$ and $U \xrightarrow{\alpha_{\mathrm{R}}} U'$.

Bisimulation equivalence is often explained in the terms of a round-based A-D game, played from an initial pair (T_0, U_0) (of regular terms) by *Attacker* (A), trying to show $T_0 \not\sim U_0$, and *Defender* (D), defending the claim $T_0 \sim U_0$; in each round, Attacker's move using (an enabled rule) $r \in \mathcal{R}$ on one side should be matched by Defender using (an enabled rule) r' on the other side, where $\textsc{act}(r') = \textsc{act}(r)$, i.e., $(r, r') \in \textsc{round}$. We do not describe the game in full; we only use some convenient notions, showing also their simple versions in $\textsc{DetCase}$.

Definition 18 *$R \subseteq \textsc{round}$ is* D-full for (T, U) *if $T \sim_1 U$ and R is a minimal set, w.r.t. the set inclusion, satisfying*
$\forall r \in \textsc{er}(T)\, \exists r' \in \textsc{er}(U) : (r, r') \in R$ and $\forall r' \in \textsc{er}(U)\, \exists r \in \textsc{er}(T) : (r, r') \in R$.

Note that there is no D-full R for (x_i, E), where $E \neq x_i$, even if $\textsc{er}(E) = \emptyset$ (since $x_i \not\sim_1 E$); on the other hand, $\emptyset$ is D-full for (x_i, x_i).
In $\textsc{DetCase}$, if $T \sim_1 U$ then there is precisely one D-full R for (T, U): for each $a \in \mathcal{A}$, where $T \xrightarrow{a}$ and $U \xrightarrow{a}$, there is precisely one $r \in \textsc{er}(T)$, $\textsc{act}(r) = a$, and precisely one $r' \in \textsc{er}(T')$, $\textsc{act}(r') = a$.
From definition of $\sim_k$ and Definition 18 we easily observe:

Observation 19 *(1.) If $\textsc{EqLv}(T, U) = k{+}1$ ($k \in \mathbb{N} \cup \{\omega\}$) then there is a D-full R for (T, U) satisfying: if $\pi \in R$ and $(T, U) \xrightarrow{\pi} (T', U')$ then $\textsc{EqLv}(T', U') \geq k$.*
(2.) If $\textsc{EqLv}(T, U) = k{+}1 < \omega$ and R is D-full for (T, U) then $\textsc{EqLv}(T', U') \leq k$ for some T', U' where $(T, U) \xrightarrow{\pi} (T', U')$, $\pi \in R$.
(3.) If $root(T) = root(T')$, $root(U) = root(U')$, and R is D-full for (T, U) then R is D-full for (T', U').

The next definition introduces D-strategies; in DETCASE we can continue after Observation 21.

Definition 20 *(1.) A nonempty prefix-closed set* $S \subseteq \text{ROUND}^{\leq d}$ *is a d-depth D-strategy for* (T,U) *if either* $d=0$ *and* $S=\{\varepsilon\}$, *or* $d>0$, $T \sim_1 U$, $R_1 = \{\pi \mid \pi \in S, |\pi|=1\}$ *is D-full for* (T,U) *and for each* $\pi \in R_1$ *we have that* $\pi\backslash S$ *is a* $(d-1)$*-depth D-strategy for* (T',U') *where* $(T,U) \xrightarrow{\pi} (T',U')$.
(2.) Given a d-depth D-strategy S for (T,U), *we define* $\text{DOM}(T,U,S) = \{(T',U') \mid (T,U) \xrightarrow{\alpha} (T',U')$ *for some* $\alpha \in S\}$ *and the* shorter domain $\text{S-DOM}(T,U,S) = \{(T',U') \mid (T,U) \xrightarrow{\alpha} (T',U')$ *for some* $\alpha \in S, |\alpha| < d\}$.

Observation 21 *If S is a d-depth D-strategy for* (T,U) *and* $(T',U') \in \text{S-DOM}(T,U,S)$ *then* $T' \sim_1 U'$.

In DETCASE, given (T,U) and $d \in \mathbb{N}$, a set S is a d-depth strategy for (T,U) iff $S = \{\alpha \in \text{ROUND}^{\leq d} \mid T \xrightarrow{\alpha_{\text{L}}}, U \xrightarrow{\alpha_{\text{R}}}\}$ and $T' \sim_1 U'$ for all $(T',U') \in \text{S-DOM}(T,U,S)$ (as defined in (2.) in Definition 20). We note the next proposition for later use; it follows easily from Observations 21 and 19(2).

Proposition 22 *Suppose a d-depth D-strategy S for* (T,U).
(1.) If $(T,U) \xrightarrow{\beta} (T',U')$ *and* $T' \xrightarrow{w}$ $(w \in \mathcal{R}^*)$ *where* $\beta \in S$, $|\beta|+|w| \leq d$, *then there is* γ *such that* $\gamma_{\text{L}} = w$ *and* $\beta\gamma \in S$.
(2.) If $\text{EQLV}(T',U') = \min\{\, \text{EQLV}(T'',U'') \mid (T'',U'') \in \text{DOM}(T,U,S)\,\} = m < \omega$, $\alpha \in S$ *and* $(T,U) \xrightarrow{\alpha} (T',U')$ *then* $|\alpha| = d$. *So* $T'' \sim_{m+1} U''$ *for each* $(T'',U'') \in \text{S-DOM}(T,U,S)$.

The next standard proposition easily follows from (inductive) definitions of $\sim_k$ and of k-depth D-strategies. The following lemma is then obvious (the existence of a k-depth D-strategy can be checked by brute force).

Proposition 23 *For* $k \in \mathbb{N}$, $\text{EQLV}(T,U) \geq k$ *iff there is a k-depth D-strategy for* (T,U).

Lemma 24 *There is an algorithm which, given* $\mathcal{G} = (\mathcal{N},\mathcal{A},\mathcal{R})$ *and an initial pair* (T_{in},U_{in}), *either terminates by computing* $\text{EQLV}(T_{in},U_{in})$ *or does not terminate in which case* $T_{in} \sim U_{in}$. *The negative problem, asking whether* $T_{in} \not\sim U_{in}$, *is thus semidecidable.*

Equivalences $\sim_k$ and $\sim$ have congruence properties wrt substitutions

Proposition 25 *(1.) If* $T \sim_k U$ *then* $E[T/x_i] \sim_k E[U/x_i]$; *if, moreover,* $E \neq x_i$ *then* $E[T/x_i] \sim_{k+1} E[U/x_i]$.
(2.) If there is no w such that $E \xrightarrow{w} x_i$ *then* $E[T/x_i] \sim E[U/x_i]$ *for any* T,U.
(3.) If $E \sim_k F$ *then* $E[V/x_i] \sim_k F[V/x_i]$.

Proof: We use inductions on k.
(1.) If $k = 0$ or $E = x_j$ (including $j = i$) then the claim is trivial; so suppose $T \sim_k U$, $k > 0$ and $root(E) = Y \in \mathcal{N}$. We note that $\{(r, r) \mid r \in \text{ER}(E)\}$ is D-full for $(E[T/x_i], E[U/x_i])$ and for each $r \in \text{ER}(E)$ we have either $E \xrightarrow{r} x_i$ and thus $(E[T/x_i], E[U/x_i]) \xrightarrow{(r,r)} (T, U)$ (where $T \sim_k U$) or $(E[T/x_i], E[U/x_i]) \xrightarrow{(r,r)} (E'[T/x_i], E'[U/x_i])$ where $E'[T/x_i] \sim_k E'[U/x_i]$ by the induction hypothesis. Hence $E[T/x_i] \sim_{k+1} E[U/x_i]$.
(2.) is obvious (we can show $\forall k : E[T/x_i] \sim_k E[U/x_i]$ by induction).
(3.) The case with $k = 0$ or $(E, F) = (x_j, x_j)$ is trivial; if one of E, F is x_j and the other is not x_j then $E \not\sim_1 F$. We thus assume $E \sim_k F$, $k > 0$, and $root(E), root(F) \in \mathcal{N}$. Take $R \subseteq \text{ROUND}$ which is D-full for (E, F) and where $E' \sim_{k-1} F'$ if $(E, F) \xrightarrow{\pi} (E', F')$, $\pi \in R$. Since R is also D-full for $(E[V/x_i], F[V/x_i])$ and $E'[V/x_i] \sim_{k-1} F'[V/x_i]$ by the induction hypothesis, we are done. □

We note an obvious generalization (of the above Point(1.)):
$V_i \sim_k V'_i$ implies $E(V_1, \ldots, V_{i-1}, V_i, V_{i+1}, \ldots, V_n) \sim_k E(V_1, \ldots, V_{i-1}, V'_i, V_{i+1}, \ldots, V_n)$.
Though generally $E(V_1, \ldots, V_{i-1}, V_i, V_{i+1}, \ldots, V_n)$ might differ from $E(V_1, \ldots, V_{i-1}, x_i, V_{i+1}, \ldots, V_n)[V_i/x_i]$ since x_i might occur in some V_j, for applying Point (1.) we can use a fresh variable x_ℓ for which we have $E(V_1, \ldots, V_{i-1}, V_i, V_{i+1}, \ldots, V_n) = E(V_1, \ldots, V_{i-1}, x_\ell, V_{i+1}, \ldots, V_n)[V_i/x_\ell]$. Such variable replacements are left implicit in the next proofs.

Later we use a sequence of subterm replacements in a given pair (T', U'), keeping the eq-level; this is captured by the next proposition (which follows easily from Propositions 25, 22(2.) and Observation 3).

Proposition 26 *Assume a d-depth D-strategy S for (T, U) and an arbitrary pair (T', U') where* $\text{EQLV}(T', U') \leq \min \{ \text{EQLV}(T'', U'') \mid (T'', U'') \in \text{DOM}(T, U, S) \}$.
If $T' = G[V/x_j]$ and (V, V') or (V', V) is in S-DOM(T, U, S)*, then* $\text{EQLV}(G[V'/x_j], U') = \text{EQLV}(T', U')$.
If $T' = G[V/x_j]$ but there is no w such that $G \xrightarrow{w} x_j$ then $\text{EQLV}(G[V'/x_j], U') = \text{EQLV}(T', U')$ *for any V'.*

A successive use of the previous proposition leads to a natural generalization where $T' = G(V_1, \ldots, V_n)$ is replaced with $T' = G(V'_1, \ldots, V'_n)$ (where $(V_i, V'_i) \in$ S-DOM(T, U, S)).

Deriving pairs $(V_n, H(V_1, \ldots, V_n))$ for substitution, getting rid of V_n

In the next section we will look at so called eq-level decreasing (n, g)-sequences of pairs in the form $(E(V_1, \ldots, V_n), F(V_1, \ldots, V_n))$. For decreasing their 'width' n, the next propositions (namely 27 and 29) are important. Point (2.) in Proposition 27 yields some $i \in \{1, 2, \ldots, n\}$; the following propositions then handle just the (technically convenient) case $i = n$, which is sufficient.

Proposition 27 *The following conditions hold for any terms* E, F, $V_1, \ldots, V_n$.
(1.) $\textsc{EqLv}(E, F) \leq \textsc{EqLv}(E(V_1, \ldots, V_n), F(V_1, \ldots, V_n))$.
(2.) If $\textsc{EqLv}(E, F) \leq k < \ell \leq \textsc{EqLv}(E(V_1, \ldots, V_n), F(V_1, \ldots, V_n))$ ($\ell \in \mathbb{N} \cup \{\omega\}$) *then there is* $i \in \{1, 2, \ldots, n\}$ *and* $\alpha \in \textsc{round}^*$ *with* $|\alpha| \leq k$ *such that* $(E, F) \xrightarrow{\alpha} (x_i, H)$, *or* $(E, F) \xrightarrow{\alpha} (H, x_i)$, *where* $H \neq x_i$ *and* $\textsc{EqLv}(V_i, H(V_1, \ldots, V_n)) \geq \ell - |\alpha| \geq \ell - k$.

Proof: (1.) By repeated application of (3.) in Proposition 25 (possibly using variable replacements).
(2.) Assume the premise and note that we cannot have $(E, F) = (x_j, x_j)$ since otherwise $E \sim F$. If $k = 0$ (so $E \not\sim_1 F$ and $E(V_1, \ldots, V_n) \sim_1 F(V_1, \ldots, V_n)$) then we cannot have $root(E), root(F) \in \mathcal{N}$; hence one of E, F is x_i while the other is not x_i, and we are done ($\alpha = \varepsilon$). If $\textsc{EqLv}(E, F) \geq 1$ then necessarily $root(E), root(F) \in \mathcal{N}$ and, by Observation 19, there is $\pi = (r, r') \in \textsc{round}$ where $r \in \textsc{er}(E)$, $r' \in \textsc{er}(F)$ and for $(E, F) \xrightarrow{\pi} (E', F')$ we have $\textsc{EqLv}(E', F') \leq k-1$ and $\ell-1 \leq \textsc{EqLv}(E'(V_1, \ldots, V_n), F'(V_1, \ldots, V_n))$. By the induction hypothesis there is α' such that $|\alpha'| \leq k-1$, $(E', F') \xrightarrow{\alpha'} (x_i, H)$, or $(E', F') \xrightarrow{\alpha'} (H, x_i)$, where $H \neq x_i$ and $\textsc{EqLv}(V_i, H(V_1, \ldots, V_n)) \geq (\ell-1) - |\alpha'| \geq (\ell-1) - (k-1) \geq \ell - k$. We then put $\alpha = \pi\alpha'$. □

Proposition 28 *If* $V_n \sim_k H(V_1, \ldots, V_n)$ *and* $H \neq x_n$ *then* $V_n \sim_k H(V_1, \ldots, V_n) \sim_k H^{lim_n}(V_1, \ldots, V_{n-1})$.

Proof: For $k = 0$ the claim holds trivially. Suppose now $k > 0$, $V_n \sim_k H(V_1, \ldots, V_n)$, $H \neq x_n$; by the induction hypothesis we have $V_n \sim_{k-1} H^{lim_n}(V_1, \ldots, V_{n-1})$. We get $H(V_1, \ldots, V_n) \sim_k H(V_1, \ldots, V_{n-1}, H^{lim_n}(V_1, \ldots, V_{n-1}))$ from (1.) in Proposition 25, and we recall $H^{lim_n}(V_1, \ldots, V_{n-1}) = H(V_1, \ldots, V_{n-1}, H^{lim_n}(V_1, \ldots, V_{n-1}))$ (as follows from Observation 6). Hence $H(V_1, \ldots, V_n) \sim_k H^{lim_n}(V_1, \ldots, V_{n-1})$. □

Proposition 29 *Assume* $\textsc{EqLv}(E(V_1, \ldots, V_n), F(V_1, \ldots, V_n)) = k$, $H \neq x_n$ *and* $\textsc{EqLv}(V_n, H(V_1, \ldots, V_n)) \geq k+1$. *For* $E' = E[H^{lim_n}/x_n]$, $F' = F[H^{lim_n}/x_n]$ *we then have* $\textsc{EqLv}(E'(V_1, \ldots, V_{n-1}), F'(V_1, \ldots, V_{n-1})) = k$.

Proof: By Proposition 28 we get $V_n \sim_{k+1} H^{lim_n}(V_1, \ldots, V_{n-1})$ and by (1.) in Proposition 25 we get $E(V_1, \ldots, V_n) \sim_{k+1} E(V_1, \ldots, V_{n-1}, H^{lim_n}(V_1, \ldots, V_{n-1})) = E'(V_1, \ldots, V_{n-1})$, and similarly $F(V_1, \ldots, V_n) \sim_{k+1} F'(V_1, \ldots, V_{n-1})$ (recall Observation 6). By Observation 3 we thus derive $\textsc{EqLv}(E'(V_1, \ldots, V_{n-1}), F'(V_1, \ldots, V_{n-1})) = k$. □

3 Decidability proof

In what follows, we assume a fixed grammar $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ if not said otherwise. (In DetCase, $\mathcal{G}$ is moreover deterministic.)

To foster intuition, we will later introduce a game between Prover (she) and Refuter (he), starting with an initial pair (T_{in}, U_{in}). Prover tries to stepwise build a finite witness (proof) for the claim $T_{in} \sim U_{in}$ while Refuter tries to invalidate this witness as long as possible. We will now explore some consequences of our previous observations, which will help Prover to force Refuter to contradict himself in finite time if (and only if) indeed $T_{in} \sim U_{in}$.

Eq-level decreasing sequences and stair-base (n, g)-sequences

A (finite) *sequence seq* of pairs $(T_0, U_0), (T_1, U_1), \ldots, (T_{\ell-1}, U_{\ell-1})$ is *eq-level decreasing* if $\omega > \text{EqLv}(T_0, U_0) > \text{EqLv}(T_1, U_1) > \cdots > \text{EqLv}(T_{\ell-1}, U_{\ell-1})$. Since $\omega > \text{EqLv}(T_0, U_0)$, the first pair yields a finite bound on $\text{LENGTH}(seq) = \ell$, namely $\ell \leq 1 + \text{EqLv}(T_0, U_0)$.

Given $n \in \mathbb{N}$ and a function $g : \mathbb{N} \to \mathbb{N}$, a (finite or infinite) sequence *seq* of pairs $(T_0, U_0), (T_1, U_1), (T_2, U_2), \ldots$ is a *stair-base sequence of width n* (with n tails) *and with g-bounded increase of (regular) heads*, called an *(n, g)-sequence* for short, if it can be presented

$$\begin{array}{lll} (T_0, U_0) = (E_0(V_1, \ldots, V_n), F_0(V_1, \ldots, V_n)), & \text{PresSize}(E_0, F_0) \leq g(0) & \\ (T_1, U_1) = (E_1(V_1, \ldots, V_n), F_1(V_1, \ldots, V_n)), & \text{PresSize}(E_1, F_1) \leq g(1) & (*) \\ (T_2, U_2) = (E_2(V_1, \ldots, V_n), F_2(V_1, \ldots, V_n)), & \text{PresSize}(E_2, F_2) \leq g(2) & \\ \ldots & & \end{array}$$

for some "common tails" $V_1, \ldots, V_n$ and "boundedly increasing heads" (E_j, F_j) satisfying $\text{PresSize}(E_j, F_j) \leq g(j)$ (for $j = 0, 1, 2, \ldots$).

Stipulating $\max \emptyset = 0$, we define the following finite number for any $s \in \mathbb{N}$:

$$\text{MaxFEL}_s = \max \{ \text{EqLv}(E, F) \mid E \not\sim F \text{ and } \text{PresSize}(E, F) \leq s \}.$$

Let us consider an (n, g)-sequence *seq* in presentation $(*)$ which is eq-level decreasing (and thus finite). Propositions 27 and 29 suggest that $\text{LENGTH}(seq)$ is bounded by a constant $k_{n,g} \in \mathbb{N}$ determined by the grammar $\mathcal{G}$. We now formalize this intuition.

For $g : \mathbb{N} \to \mathbb{N}$ and $\text{MEL} \in \mathbb{N}$ we define the function $wd(g, \text{MEL}) : \mathbb{N} \to \mathbb{N}$ ('wd' for 'width-decreasing', 'MEL' for 'maximal eq-level'):

$$wd(g, \text{MEL})(j) = g(1 + \text{MEL} + j) + 2 \cdot (g(0) + \text{SizeInc} \cdot \text{MEL}) \tag{4}$$

Proposition 30 *Every eq-level decreasing (n, g)-sequence seq in presentation $(*)$, where*

$$n > 0,\ \text{EqLv}(E_0, F_0) \leq \text{MEL},\ \text{LENGTH}(seq) = (1 + \text{MEL}) + \ell,\ \ell \geq 1,$$

implies the existence of an eq-level decreasing $(n-1, wd(g, \text{MEL}))$-sequence seq' starting with some (T'_0, U'_0) where $\text{EqLv}(T'_0, U'_0) \leq \text{EqLv}(T_0, U_0) - (1 + \text{MEL})$ and $\text{LENGTH}(seq') = \ell$.

Proof: Since $\text{EqLv}(E_0, F_0) \leq \text{EqLv}(T_0, U_0)$ (by (1.) in Proposition 27) and $\text{LENGTH}(seq) > 1 + \text{MEL} \geq 1 + \text{EqLv}(E_0, F_0)$, we must have $\text{EqLv}(E_0, F_0) < \text{EqLv}(T_0, U_0)$. By Proposition 27(2.) there is $i \in \{1, 2, \ldots, n\}$ and $H \neq x_i$ where $E_0 \xrightarrow{w} H$ or $F_0 \xrightarrow{w} H$ for some $w \in \mathcal{R}^*$, $|w| \leq \text{EqLv}(E_0, F_0)$, such that $\text{EqLv}(V_i, H(V_1, \ldots, V_n)) \geq \text{EqLv}(T_0, U_0) - |w|$. We have $\text{PresSize}(H) \leq g(0) + \text{SizeInc} \cdot \text{MEL}$ (by Observation 10(2)).

Wlog we can assume $i = n$ (achieved by swapping x_i with x_n and V_i with V_n). Proposition 29 guarantees that for all j, $\textsc{mel}+1 \leq j \leq \textsc{mel}+\ell$ we have, denoting $\Delta = \textsc{mel}+1$: $\textsc{EqLv}(E'_{j-\Delta}(V_1,\ldots,V_{n-1}), F'_{j-\Delta}(V_1,\ldots,V_{n-1})) = \textsc{EqLv}(E_j(V_1,\ldots,V_n), F_j(V_1,\ldots,V_n))$ where $E'_{j-\Delta} = E_j[H^{lim_n}/x_n]$ and $F'_{j-\Delta} = F_j[H^{lim_n}/x_n]$. For $j = 0,1,\ldots,\ell{-}1$ we thus have $\textsc{PresSize}(E'_j, F'_j) \leq \textsc{PresSize}(E_{j+\Delta}, F_{j+\Delta}) + 2\cdot\textsc{PresSize}(H) \leq$

$$\leq g(1+\textsc{mel}+j) + 2\cdot(g(0)+\textsc{SizeInc}\cdot\textsc{mel}) = wd(g,\textsc{mel})(j).$$

We thus get an eq-level decreasing $(n{-}1, wd(g,\textsc{mel}))$-sequence satisfying the claim. □

Nondeterministical bounding of eq-level decreasing stair-base sequences

Let us consider the following nondeterministic (recursive) procedure

Bound(n,g): (* $n \in \mathbb{N}$, g is (a Turing machine for) a computable function *)
$\textsc{mel} := \textsc{MaxFEL}'_{g(0)}$;
if $n=0$ then $k'_{n,g} := 1+\textsc{mel}$ else $k'_{n,g} := (1+\textsc{mel}) + \mathit{Bound}(n{-}1, wd(g,\textsc{mel}))$;
return $k'_{n,g}$

Computing $\textsc{MaxFEL}'_{g(0)}$ is nondeterministic: Enumerate (graph presentations of) all pairs (E,F), $\textsc{PresSize}(E,F) \leq g(0)$; for each such (E,F) either compute the finite $\textsc{EqLv}(E,F)$ (recall Lemma 24) or include (E,F) into a global variable $\textsc{GuessEq}$, which collects the pairs which are guessed bisimulation equivalent; the maximum of the computed finite eq-levels (or 0) is taken as $\textsc{MaxFEL}'_{g(0)}$. ($\textsc{MaxFEL}'_{g(0)}$ is thus a lower bound for $\textsc{MaxFEL}_{g(0)} = \max\{\, \textsc{EqLv}(E,F) \mid E \not\sim F \text{ and } \textsc{PresSize}(E,F) \leq g(0)\,\}$.)

So each run of *Bound*(n,g) comprises $n{+}1$ calls of *Bound*, namely *Bound*(n,g_n), *Bound*$(n{-}1,g_{n-1})$, …, *Bound*$(0,g_0)$, and computes successively some values

$\textsc{MaxFEL}'_{g_n(0)}$, $\textsc{MaxFEL}'_{g_{n-1}(0)}$, …, $\textsc{MaxFEL}'_{g_0(0)}$, k'_{0,g_0}, k'_{1,g_1}, …, k'_{n,g_n}, and $\textsc{GuessEq}$,

where $g_n = g$ and $g_i = wd(g_{i+1}, \textsc{MaxFEL}'_{g_{i+1}(0)})$ for $i = n{-}1, n{-}2, \ldots, 0$; $\textsc{GuessEq}$ contains all pairs guessed equivalent during the run. We refer to these values in the next proposition.

Proposition 31 *Given a run of Bound(n,g), let $m = \min\{\textsc{EqLv}(E,F) \mid (E,F) \in \textsc{GuessEq})\}$ ($m \in \mathbb{N}\cup\{\omega\}$). For each $i \in \{0,1,\ldots,n\}$, there is no eq-level decreasing (i,g_i)-sequence seq starting with (T_0,U_0) where $\textsc{EqLv}(T_0,U_0) < m$ and $\textsc{length}(seq) > k'_{i,g_i}$.*

Proof: Assuming the claim holds for $0,1,\ldots,i{-}1$, we consider an eq-level decreasing (i,g_i)-sequence *seq* in presentation $(*)$ (with $n=i$), starting with a pair $(T_0,U_0) = (E_0(V_1,\ldots,V_i), F_0(V_1,\ldots,V_i))$ where $m > \textsc{EqLv}(T_0,U_0) \geq \textsc{EqLv}(E_0,F_0)$. Hence $(E_0,F_0) \notin \textsc{GuessEq}$ and the (sub)run *Bound*(i,g_i) has computed $\textsc{EqLv}(E_0,F_0)$ (since $\textsc{PresSize}(E_0,F_0) \leq g_i(0)$); thus $\textsc{EqLv}(E_0,F_0) \leq \textsc{MaxFEL}'_{g_i(0)}$. If $i=0$ then we get $\textsc{length}(seq) \leq 1 + \textsc{MaxFEL}'_{g_0(0)} = k'_{0,g_0}$. If $i>0$ then Proposition 30 and our (induction) assumption guarantees that $\textsc{length}(seq) \leq (1+\textsc{MaxFEL}'_{g_i(0)}) + k'_{i-1,g_{i-1}} = k'_{i,g_i}$. □

Corollary 32 *For every (n,g) there is the least $k_{n,g} \in \mathbb{N}$ such that each eq-level decreasing (n,g)-sequence has length at most $k_{n,g}$.*

Prover-Refuter game

Given $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ and an initial pair (T_{in}, U_{in}), the game is played as follows.

1. Prover chooses $n \in \mathbb{N}$ and a Turing machine (claimed to be) computing a function $g : \mathbb{N} \to \mathbb{N}$, and performs a chosen run of *Bound*(n, g), returning $k'_{n,g}$ and GuessEq. (In fact, later we show some concrete n and g which can be computed from the grammar $\mathcal{G}$, independently of the initial pair.)

2. Refuter chooses $(T_0, U_0) \in$ StartSet where StartSet $= \{(T_{in}, U_{in})\} \cup$ GuessEq. Refuter claims that EqLv$(T_0, U_0) = \min\{$ EqLv$(T, U) \mid (T, U) \in$ StartSet $\} < \omega$.

3. Now the following *phases* are performed repeatedly: the i-th phase ($i = 0, 1, 2, \ldots$) starts with a pair (T_i, U_i) and produces some (T_{i+1}, U_{i+1}) unless a player wins in this phase (in which case the play finishes):

   (a) Prover chooses a d-depth D-strategy S_i from (T_i, U_i), for some $d > 0$. If not possible (since $T_i \not\sim_1 U_i$) then Refuter wins. If for each maximal $\alpha \in S_i$ we have $(T_i, U_i) \xrightarrow{\alpha} (T, U)$ where $T \sim_1 U$ (in $LTS^{\mathrm{A}}_{\mathcal{G}}$) and both T, U are dead (in $LTS^{\mathrm{R}}_{\mathcal{G}}$, so we can have $(T, U) = (x_j, x_j)$) then Prover wins.

   (b) Refuter chooses $\alpha_i \in S_i$ with $|\alpha_i| = d$; let $(T_i, U_i) \xrightarrow{\alpha_i} (T'_i, U'_i)$. Refuter claims that EqLv$(T'_i, U'_i) = \min\{$ EqLv$(T, U) \mid (T, U) \in$ Dom$(T_i, U_i, S_i) \}$. (Recall 2. in Proposition 22.)

   (c) Prover produces (T_{i+1}, U_{i+1}) from (T'_i, U'_i) by a (maybe empty) sequence of subterm replacements captured by Proposition 26, using pairs $(V, V') \in$ S-Dom(T_i, U_i, S_i); this guarantees EqLv$(T_{i+1}, U_{i+1}) =$ EqLv(T'_i, U'_i) if Refuter's claim in 3.b is true.

   (d) If there is a repeat, i.e. $(T_{i+1}, U_{i+1}) = (T_j, U_j)$ for some $j \leq i$, Prover wins.

   (e) Prover also wins if (T_0, U_0), (T_1, U_1), (T_2, U_2), ..., (T_{i+1}, U_{i+1}) contains (is shown to contain) a subsequence *seq* $= (T_{i_1}, U_{i_1})$, (T_{i_2}, U_{i_2}), (T_{i_3}, U_{i_3}), ... with $0 < i_1 < i_2 < i_3 < \ldots$ which is an (n, g)-sequence and length$(seq) > k'_{n,g}$.

We could surely allow also other (sound) possibilities for Prover (e.g., Prover wins when $(T_i, U_i) = (E(V_1, \ldots, V_n), F(V_1, \ldots, V_n))$ where $i \geq 1$ and $(E, F) \in$ GuessEq) but the above mentioned game-rules turn out sufficient (for completeness).

Lemma 33 *It is semidecidable, given* $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ *and an initial pair* (T_{in}, U_{in})*, if Prover has a winning strategy (forcing her win in finite time). If Prover has a winning strategy using* GuessEq *then* $T \sim U$ *for all* $(T, U) \in \{(T_{in}, U_{in})\} \cup$ GuessEq.

Proof: Semidecidability follows easily from the fact that Refuter always has only finitely many choices when there is his turn. The rest (i.e. soundness) follows from Proposition 31: if $\min\{$ EqLv$(T, U) \mid (T, U) \in \{(T_{in}, U_{in})\} \cup$ GuessEq $\} < \omega$ then Refuter can be choosing so that his 'least eq-level' claims (in 2. and 3.b) are always true, and Prover cannot win (it is Refuter who can force his win in finite time in this case). □

A winning strategy for Prover in the case $T_{in} \sim U_{in}$

Suppose that Refuter has chosen (T_0, U_0) (in step 2) where $T_0 \sim U_0$, but we now ignore the Prover's winning condition 3(e). We will show a concrete (though still nondeterministic) Prover's strategy by which she will not lose but the run of the game can be infinite. Examining the strategy in detail, we then discover that an infinite (n, g)-sequence would thus arise, for concrete n and g computable from $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$; hence Prover can indeed force her win in the presence of 3(e).

We concretize the game phases so that they produce not just pairs (T_i, U_i) ($i = 0, 1, 2 \ldots$), but tuples $(T_i, U_i, side_i)$ where $side_i \in \{\text{L}, \text{R}, nil\}$. As will be clarified, $side_i$ serves Prover to remember the side of the *resthead* from the balancing step in the previous phase if there is such a resthead. Recall now constants DINC and M_0 (defined by (3)); we also recall the instance of Proposition 14 for $\ell_0 = M_0$ and $\ell_1 = M_1$ where

$$M_1 = M_0 \cdot (3 + 2M_0 \cdot \text{DINC}). \tag{5}$$

We start with (T_0, U_0, nil) (asssuming $T_0 \sim U_0$).

In the phase starting with $(T_i, U_i, side_i)$:

1. Prover chooses an M_1-depth D-strategy S_i from (T_i, U_i), so that $T \sim U$ for all $(T, U) \in \text{DOM}(T_i, U_i, S_i)$.
2. Refuter chooses $\alpha_i \in S_i$, $|\alpha_i| = M_1$, if any exists; if not, Prover wins (by dead pairs). Let $(T_i, U_i) \xrightarrow{\alpha_i} (T'_i, U'_i)$.
3. (a) If $side_i = \text{L}$ then:
   - If there is a finite term G, $\text{DEPTH}(G) \leq 2 + 2M_0 \cdot \text{DINC}$, such that T'_i can be presented as $T'_i = G(V_1, \ldots, V_m)$ where for each V_j ($1 \leq j \leq m$) there is V'_j such that $(V_j, V'_j) \in \text{S-DOM}(T_i, U_i, S_i)$ or $V'_j = U_i$ when there is no w such that $G \xrightarrow{w} x_j$, then Prover chooses one such *resthead* G with the least $\text{DEPTH}(G)$, and performs a *left balancing step*, i.e. puts $(T_{i+1}, U_{i+1}, side_{i+1}) = (G(V'_1, \ldots, V'_m), U'_i, side_{i+1})$ where $side_{i+1} = nil$ if G is just a variable (so if $(T'_i, V') \in \text{S-DOM}(T_i, U_i, S_i)$ for some V') and $side_{i+1} = \text{L}$ otherwise.
   - If there is no such G, Prover puts $(T_{i+1}, U_{i+1}, side_{i+1}) = (T'_i, U'_i, nil)$.

   (b) If $side_i = \text{R}$ then Prover behaves symmetrically to the case (a), performing a *right balancing step* if possible.

   (c) If $side_i = nil$ then: if a left or right balancing step is possible, Prover performs a chosen one (defining $(T_{i+1}, U_{i+1}, side_{i+1})$ appropriately); otherwise she puts $(T_{i+1}, U_{i+1}, side_{i+1}) = (T'_i, U'_i, nil)$.

For a left balancing step as described above, U_i is the *(balancing) pivot* and the pair $(T_{i+1}, U_{i+1}) = (G(V'_1, \ldots, V'_m), U'_i)$ is the *balancing result.* Symmetrically, a right balancing step has the pivot T_i and the result $(T'_i, G(V'_1, \ldots, V'_m))$.

Proposition 34 *If U_i (or T_i) is a pivot and $(G(V'_1,\ldots,V'_m),U'_i)$ (or $(T'_i,G(V'_1,\ldots,V'_m))$) is the respective balancing result then all $V'_1,\ldots,V'_m$ and U'_i (or T'_i) are reachable from the pivot U_i (or T_i) by words of length $\leq M_1$ (and $\text{DEPTH}(G) \leq 2+2M_0\cdot\text{DINC}$). The number of balancing results with one pivot is thus (boundedly) finite.*

Proposition 35 *Let us have (T_i,U_i), S_i,α_i, $(T_i,U_i)\xrightarrow{\alpha_i}(T'_i,U'_i)$, as above, and assume that a left balancing step is not possible. Then (T_i sinks and the possible resthead is erased):*
(1.) $T_i \xrightarrow{(\alpha_i)_{\text{L}}} T'_i$ is M_0-sinking.
(2.) If ($side_i = \text{L}$ and) $(T_i,U_i)=(G'(W_1,\ldots,W_{m'}),U_i)$, where G' is the resthead from the previous phase (with pivot U_{i-1}) then $G'\xrightarrow{u} x_j$ for a prefix u of α_{L} (so $T_i\xrightarrow{u}W_j\xrightarrow{u'}T'_i$ where $\alpha_{\text{L}}=uu'$) and thus both T'_i,U'_i are reachable from the previous pivot (U_{i-1} in our case $side_i=\text{L}$) by words of length at most $2M_1$.

Proof: (1.) Suppose $T_i\xrightarrow{(\alpha_i)_{\text{L}}}T'_i$ is not M_0-sinking. Then we have the longest β such that $\alpha_i=\beta\gamma\delta$, $|\gamma|=M_0$, and $Yx_1\ldots x_m\xrightarrow{\gamma_{\text{L}}}F$ (for some Y,F) where
$(T_i,U_i)\xrightarrow{\beta}(YV_1\ldots V_m,U')\xrightarrow{\gamma}(F(V_1,\ldots,V_m),U'')\xrightarrow{\delta}(T'_i,U'_i)$.

Note that for each $j, 1\leq j\leq m$, we either have $\text{SSW}(Y,j)=\varepsilon$ (there is no (Y,j)-sink word) or there is γ_j such that $(\gamma_j)_{\text{L}}=\text{SSW}(Y,j)$ and $\beta\gamma_j\in S_i$ (note $|\gamma_j|=|\text{SSW}(Y,j)|<M_0$ and recall 1. in Proposition 22), in which case we have $(T_i,U_i)\xrightarrow{\beta}(YV_1\ldots V_m,U')\xrightarrow{\gamma_j}(V_j,V'_j)$ for some V'_j; thus $(V_j,V'_j)\in\text{S-DOM}(T_i,U_i,S_i)$.

If $\delta=\delta'\delta''$ where $F\xrightarrow{\delta'_{\text{L}}}x_j$ then $V_j\xrightarrow{(\delta'')_{\text{L}}}T'_i$ and thus also $(T_i,U_i)\xrightarrow{\beta}(YV_1\ldots V_m,U')\xrightarrow{\gamma_j}(V_j,V'_j)\xrightarrow{\mu}(T'_i,V''_j)$ for some μ such that $\beta\gamma_j\mu\in S_i$ and $\mu_{\text{L}}=\delta''_{\text{L}}$ (by Proposition 22(1)). Hence $(T'_i,V''_j)\in\text{S-DOM}(T_i,U_i,S_i)$, and a left-balancing was possible.

Hence we have $Yx_1\ldots x_m\xrightarrow{\gamma_{\text{L}}}F\xrightarrow{\delta_{\text{L}}}G$ for some G; the segment $F\xrightarrow{\delta_{\text{L}}}G$ must be M_0-sinking (otherwise β was not the longest). We thus derive
$\text{DEPTH}(G)\leq 2+M_0\cdot\text{DINC}+(M_0-1)\cdot\text{DINC}\leq 2+2\cdot M_0\cdot\text{DINC}$. So there is a left-balancing step, resulting in $(G(V'_1,\ldots,V'_m),U'_i)$ – a contradiction.

(2.) now follows easily, using Proposition 14(1) (for $\ell_0=M_0$, $\ell_1=M_1$). □

Lemma 36 *If Prover uses the above strategy from (T_0,U_0), $T_0\sim U_0$, (chosen by Refuter in the game-step 2.), but does not use 3.(e), then she either wins by dead-pairs or by a repeat, or the sequence $(T_1,U_1),(T_2,U_2),(T_3,U_3),\ldots$ contains an infinite (n,g)-sequence, for some n,g computable from $\mathcal{G}$.*

Proof: Assume a play gives rise to an infinite sequence $(T_0,U_0),(T_1,U_1),(T_2,U_2),\ldots$.
If there were only finitely many balancing steps then for some $i\geq 0$ we had
$(T_i,U_i)\xrightarrow{\alpha_i}(T_{i+1},U_{i+1})\xrightarrow{\alpha_{i+1}}(T_{i+2},U_{i+2})\xrightarrow{\alpha_{i+2}}\cdots$ where for $\mu=\alpha_i\alpha_{i+1}\alpha_{i+2}\ldots$ both $T_i\xrightarrow{\mu_{\text{L}}}$ and $U_i\xrightarrow{\mu_{\text{R}}}$ range over finitely many terms (by Proposition 14(2), for $\ell_0=M_0$, $\ell_1=M_1$); we would thus get a repeat. Hence there is a sequence $i_1<i_2<i_3<\cdots$ where for each $j=1,2,3,\ldots$ either U_{i_j} or T_{i_j} is a balancing pivot B_j. Moreover, we cannot have that

B_j are the same for infinitely many j, since otherwise the balancing results would yield a repeat (by Proposition 34).

In our above discussion (Propositions 34 and 35) we have, in fact, shown a *pivot path* $B_1 \xrightarrow{w_1} B_2 \xrightarrow{w_2} B_3 \xrightarrow{w_3} \ldots$ in the form of Proposition 15 ($\ell_0 = M_0$, $\ell_1 = M_1$) where w_j arises from $(\alpha_{i_j}\alpha_{i_j+1}\ldots\alpha_{i_{j+1}-1})_{side}$, where $side \in \{\textsc{l},\textsc{r}\}$, by a possible replacing of a prefix of length $\leq 2M_1$ by some shorter word (in the case B_j and B_{j+1} are opposite-side pivots). We thus must have

$$B_1 \xrightarrow{uv_1} YU_1\ldots U_m \xrightarrow{v_2} H_0(U_1,\ldots,U_m) \xrightarrow{w_{k+1}} H_1(U_1,\ldots,U_m) \xrightarrow{w_{k+2}} H_2(U_1,\ldots,U_m) \xrightarrow{w_{k+3}} \cdots$$

where H_j are finite terms with $\textsc{Depth}(H_j) \leq g_{M_1}(j)$ as in the case 2. of Proposition 15.

Let $V_1,\ldots,V_n$ comprise all M_1-depth subterms of $V = YU_1\ldots U_m$ and all variables occurring in V in depth less than M_1; we have $n \leq c^{M_1}$ where $c = \max\{\, arity(Y) \mid Y \in \mathcal{N}\,\}$. Recalling Observation 10(3), we note that each pivot $H_j(U_1,\ldots,U_m)$ can be presented as $H'_j(V_1,\ldots,V_n)$ where $\textsc{Depth}(H'_j) \leq \textsc{Depth}(H_j) + M_1$ and where $H_j(U_1,\ldots,U_m) \xrightarrow{w} W$, $|w| \leq M_1$, implies $W = H'(V_1,\ldots,V_n)$ where $H'_j \xrightarrow{w} H'$.

Recalling Proposition 34, we deduce that the balancing result for pivot $H_j(U_1,\ldots,U_m)$ ($j = 0,1,2,\ldots$) can be presented as $(E_j(V_1,\ldots,V_n), F_j(V_1,\ldots,V_n))$, where E_j, F_j are finite terms with $\textsc{Depth}(E_j)$, $\textsc{Depth}(F_j)$ bounded by $(g_{M_1}(j){+}M_1){+}M_1{\cdot}\textsc{Dinc}{+}(2{+}2M_0{\cdot}\textsc{Dinc})$. This obviously yields a computable $g : \mathbb{N} \to \mathbb{N}$ such that $\textsc{PresSize}(E_j, F_j) \leq g(j)$. $\Box$

Lemmas 33 and 36 easily yield the semidecidability for $T_{in} \overset{?}{\sim} U_{in}$; the next theorem thus follows by recalling Lemma 24.

Theorem 37 *Bisimulation equivalence for first-order grammars is decidable.*

Additional remarks

Reducing the bisimilarity problem for nondeterministic pushdown automata solved in [3] to first-order grammars is straightforward: pushdown configurations can be naturally transformed to terms, pushdown rules to term root-rewriting rules, and the deterministic ε-popping steps can be 'precomputed' by a natural pruning of the terms (cf. [1]).

In the deterministic case (with at most one rule $r : Yx_1\ldots x_m \xrightarrow{a} E$ for each pair (Y,a)) there is no D-strategy to choose (since it is unique) and the eq-level of a pair (T,U) can drop by at most one in one step. Prover can derive the relevant pairs $(V_n, H^{lim_n}(V_1,\ldots,V_{n-1}))$ and perform the appropriate replacements directly. The set $\textsc{GuessEq}$ can be seen as a finite *basis* (computed from the grammar $\mathcal{G}$). Prover does not need to count; a sufficient winning condition is getting $(T_i,U_i) = (E(V_1,\ldots,V_n), F(V_1,\ldots,V_n))$ where $i \geq 1$ and $(E,F) \in \textsc{GuessEq}$.

There is a natural question of the relation of the proof presented here to the proof by Sénizergues [3]. This is difficult to answer, since it would require a detailed technical analysis of [3]. Looking *retrospectively*, it is well possible that one could find all abstract ideas used here embedded in some form in [3] and then start to view the proof presented here as a sort of 'translation' of the original proof. (A neutral reader is encouraged to try to do this.) In fact, many ideas preceded [3]. E.g., it seems that Sénizergues gives credit to Meitus for an analogue of 2. in Proposition 27 (in the simpler deterministic case), referring to a published paper by Meitus which claims to solve the dpda language equivalence. (Btw, Sénizergues does not explain what was Meitus' principal mistake and how he corrected this; Meitus seems to continue complaining, viewing himself as the one who solved the problem. Such a question can be also hardly clarified without a detailed analysis, which seems very difficult due to the form of Meitus' paper.)

Acknowledgements

I am grateful to Colin Stirling for initial discussions which helped me to start to build the intuition finally leading to the proof presented here. I thank Arnaud Carayol and Didier Caucal for valuable comments on [1]. I also thank Philippe Schnoebelen and LSV ENS Cachan for organizing http://www.lsv.ens-cachan.fr/Events/Pavas/; though it was a bit premature, I hope that it was useful not only for me.

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