Tutorial 6 - Solutions

Exercise 1*

Consider the set $\{a, b, c\}$ (with three elements). Define some nontrivial function $f : 2^{\{a,b,c\}} \rightarrow 2^{\{a,b,c\}}$ which is monotonic.

For example, we can define f as follows (note that there are many possibilites):

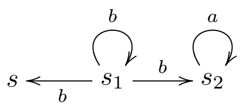
S	$f(S)$
$\emptyset$	$\{a\}$
$\{a\}$	$\{a\}$
$\{b\}$	$\{a\}$
$\{c\}$	$\{a\}$
$\{a, b, c\}$	$\{a, b\}$
$\{a, b\}$	$\{a, b\}$
$\{a, c\}$	$\{a, b\}$
$\{b, c\}$	$\{a, b\}$

The function f is monotonic which we can verify by a case inspection.

- Compute the greatest fixed point by using directly the Tarski's fixed point theorem.
  - According to Tarski's fixed point theorem the greatest fixed point z_{max} is given by $z_{\text{max}} = \cup A$, where
$$A = \{x \in 2^{\{a,b,c\}} \mid x \subseteq f(x)\}.$$
    In our case, by the definition of f we get $A = \{\emptyset, \{a\}, \{a, b\}\}$. The union of the sets in A is $\{a, b\}$, so by Tarski's fixed point theorem, the greatest fixed point of f is $\{a, b\}$.
- Compute the least fixed point of f by starting from $\emptyset$ and applying repeatedly the function f until the fixed point is reached.
$$f(\emptyset) = \{a\}$$
$$f(f(\emptyset)) = f(\{a\}) = \{a\}$$
  Hence the least fixed point of f is $\{a\}$.

Exercise 2

Consider the following labelled transition system.

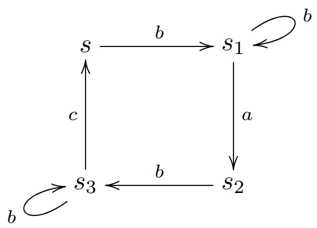


Compute for which sets of states $[\![X]\!] \subseteq \{s, s_1, s_2\}$ the following formulae are true.

- $X = \langle a\rangle tt \vee [b]X$
  - The equation holds for the following sets of states: $\{s_2, s\}, \{s_2, s_1, s\}$.
- $X = \langle a\rangle tt \vee ([b]X \wedge \langle b\rangle tt)$
  - The equation holds only for the set $\{s_2\}$.

Exercise 3*

Consider the following labelled transition system.



Using the game characterization for recursive Hennessy-Milner formulae decide whether the following claims are true or false and discuss what properties the formulae describe:

- $s \models X$ where $X \overset{\min}{=} \langle c \rangle \mathit{tt} \vee \langle Act \rangle X$
  - A universal winning strategy for the defender starting from (s, X) is as follows:

$$\begin{aligned} (s, X) &\rightarrow (s, \langle c \rangle \mathit{tt} \vee \langle Act \rangle X) \xrightarrow{D} (s, \langle Act \rangle X) \xrightarrow{D} (s_1, X) \\ &\rightarrow (s_1, \langle c \rangle \mathit{tt} \vee \langle Act \rangle X) \xrightarrow{D} (s_1, \langle Act \rangle X) \xrightarrow{D} (s_2, X) \\ &\rightarrow (s_2, \langle c \rangle \mathit{tt} \vee \langle Act \rangle X) \xrightarrow{D} (s_2, \langle Act \rangle X) \xrightarrow{D} (s_3, X) \\ &\rightarrow (s_3, \langle c \rangle \mathit{tt} \vee \langle Act \rangle X) \xrightarrow{D} (s_3, \langle c \rangle \mathit{tt}) \xrightarrow{D} (s, \mathit{tt}), \end{aligned}$$

    where (s, tt) by definition is a winning configuration for the defender.

- $s \not\models X$ where $X \overset{\min}{=} \langle c \rangle \mathit{tt} \vee [Act] X$
  - A universal winning strategy for the attacker is as follows: $(s, X) \rightarrow (s, \langle c \rangle \mathit{tt} \vee [Act] X)$ Then if the defender plays $\langle c \rangle \mathit{tt}$, he loses since there are no c-transitions from s, thus the defender must play $(s, \langle c \rangle \mathit{tt} \vee [Act] X) \xrightarrow{D} (s, [Act] X)$. Then the attacker plays $(s, [Act] X) \xrightarrow{A} (s_1, X)$. And we have $(s_1, X) \rightarrow (s_1, \langle c \rangle \mathit{tt} \vee [Act] X)$. Now for similar reasons as above the defender must choose to play $(s_1, \langle c \rangle \mathit{tt} \vee [Act] X) \xrightarrow{D} (s_1, [Act] X)$. The attacker plays $(s_1, [Act] X) \xrightarrow{A} (s_1, X)$ which is a configuration we have seen earlier. Thus either the play is infinite, in which case the attacker wins since X is defined as the least fixed-point. Or the play is finite, in which case the attacker also wins.

- $s \models X$ where $X \overset{\max}{=} \langle b \rangle X$
  - A universal winning strategy for the defender is:

$$(s, X) \rightarrow (s, \langle b \rangle X) \xrightarrow{D} (s_1, X) \rightarrow (s_1, \langle b \rangle X) \xrightarrow{D} (s_1, X).$$

    Thus the play is infinite, and since X is defined as the greatest fixed-point, the defender wins.

- $s \models X$ where $X \overset{\max}{=} \langle b \rangle \mathit{tt} \wedge [a] X \wedge [b] X$
  - Universal winning strategy for the defender: We have $(s, X) \rightarrow (s, \langle b \rangle \mathit{tt} \wedge [a] X \wedge [b] X)$. Now if the attacker plays $(s, \langle b \rangle \mathit{tt} \wedge [a] X \wedge [b] X) \xrightarrow{A} (s, \langle b \rangle \mathit{tt})$ he loses since the defender can then play $(s, \langle b \rangle \mathit{tt}) \xrightarrow{D} (s_1, \mathit{tt})$. Furthermore if the attacker plays $(s, \langle b \rangle \mathit{tt} \wedge [a] X \wedge [b] X) \xrightarrow{A} (s, [a] X)$, then he also loses since he is stuck in the configuration $(s, [a] X)$. The third option for the attacker is to choose $(s, \langle b \rangle \mathit{tt} \wedge [a] X \wedge [b] X) \xrightarrow{A} (s, [b] X) \xrightarrow{A} (s_1, X)$.
    Expanding X we get $(s_1, X) \rightarrow (s_1, \langle b \rangle \mathit{tt} \wedge [a] X \wedge [b] X)$. From here if the attacker plays $(s_1, \langle b \rangle \mathit{tt} \wedge [a] X \wedge [b] X) \xrightarrow{A} (s_1, \langle b \rangle \mathit{tt})$ he loses since the defender can play $(s_1, \langle b \rangle \mathit{tt}) \xrightarrow{D}$

(s_1, tt). If the attacker plays $(s_1, \langle b \rangle tt \wedge [a]X \wedge [b]X) \xrightarrow{A} (s_1, [b]X)$, then the only possible next move is $(s_1, [b]X) \xrightarrow{A} (s_1, X)$ which is a previously encountered configuration. The last option for the attacker is to play $(s_1, \langle b \rangle tt \wedge [a]X \wedge [b]X) \xrightarrow{A} (s_1, [a]X) \xrightarrow{A} (s_2, X)$.
Expanding the encoding we get $(s_2, X) \rightarrow (s_2, \langle b \rangle tt \wedge [a]X \wedge [b]X)$. Again if the attacker plays $(s_2, \langle b \rangle tt \wedge [a]X \wedge [b]X) \xrightarrow{A} (s_2, \langle b \rangle tt)$ he loses by the defenders move $(s_2, \langle b \rangle tt) \xrightarrow{D} (s_3, tt)$. If the attacker plays $(s_2, \langle b \rangle tt \wedge [a]X \wedge [b]X) \xrightarrow{A} (s_2, [a]X)$ he loses since he is stuck. Finally he can play $(s_2, \langle b \rangle tt \wedge [a]X \wedge [b]X) \xrightarrow{A} (s_2, [b]X) \xrightarrow{A} (s_3, X)$.
Expanding X we obtain $(s_3, X) \rightarrow (s_3, \langle b \rangle tt \wedge [a]X \wedge [b]X)$. Now playing $(s_3, \langle b \rangle tt \wedge [a]X \wedge [b]X) \xrightarrow{A} (s_3, \langle b \rangle tt)$ he loses by the defenders move $(s_3, \langle b \rangle tt) \xrightarrow{D} (s_3, tt)$. If the attacker plays $(s_3, \langle b \rangle tt \wedge [a]X \wedge [b]X) \xrightarrow{A} (s_3, [a]X)$ he is stuck. Finally the attacker can play $(s_3, \langle b \rangle tt \wedge [a]X \wedge [b]X) \xrightarrow{A} (s_3, [b]X) \xrightarrow{A} (s_3, X)$ which is a previously encountered configuration.
Thus either the attacker loses in a finite play, or the play is infinite in which case the defender wins since X is defined as the greatest fixed-point.