Decidability of Bisimilarity for One-Counter Processes[1]

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Abstract.

It is shown that bisimulation equivalence is decidable for the processes generated by (nondeterministic) pushdown automata where the pushdown behaves like a counter. Also finiteness up to bisimilarity is shown to be decidable for the mentioned processes.

1 Introduction

In recent years, a growing effort has been devoted to the area of verification of (potentially) infinite-state systems. An important studied question is that of (un)decidability for various behavioural equivalences. A prominent role among these equivalences is played by *bisimulation equivalence*, or *bisimilarity*, which is more appropriate for reactive systems than e.g. the traditional language equivalence (Milner, 1989).

Several recent results help to highlight and understand the decidability boundaries for bisimilarity, which are different from those for language equivalence. It is e.g. known that bisimilarity is decidable for Basic Parallel Processes, BPP (Christensen, Hirshfeld, and Moller, 1993), while language equivalence is undecidable for them (Hirshfeld, 1994). More relevant here are context-free processes (generated by context-free grammars), also called BPA-processes, where language equivalence is well-known to be undecidable while bisimilarity is decidable (Christensen, Hüttel, and Stirling, 1995). Pushdown automata (which are in the 'language sense' equivalent to context-free grammars) generate a richer family than that of context-free processes when considering bisimulation equivalence. The *pushdown processes* can be identified with 'state-stack' configurations, whose behaviour is determined by the transition rules (not allowing ε-rules). Stirling (1996) has shown decidability of bisimilarity for *normed* pushdown processes while the question for the whole class has been left open (*normedness* means that any reachable configuration can evolve into a configuration with the empty stack).

Here we show the decidability of bisimilarity for another subclass of pushdown processes: we do not impose the restriction of normedness but we consider the case when the stack behaves like a counter; i.e. there is only one stack symbol—besides a special bottom (or zero) symbol which enables to test 'emptiness' of the stack (i.e. to test whether the counter is zero). Let us call such processes *one-counter processes*. The decidability result for one-counter processes also confirms the conjecture by the author (Jančar, 1993) that bisimilarity for labelled Petri nets with one unbounded place is decidable (while two unbounded places suffice for undecidability); it follows from the fact that there is a straightforward transformation of a net with one unbounded place into a one-counter machine—each reachable (sub)marking on the bounded places corresponds to a control state of the machine. (Note that such a machine does not test for zero.)

It should be mentioned here that G. Sénizergues announced (in November 1997) that the technique of his outstanding result (Sénizergues, 1997) generalizes to bisimulation equivalence for nondeterministic pushdown automata; then the main results of (Stirling, 1996) and of this paper follow from the announced general result. Nevertheless due to the extreme length and complexity of the general proof (which still has to be verified), the presented proof for one-counter processes should retain its value.

Semidecidability of *nonbisimilarity* of one-counter processes (as well as pushdown processes) can be derived easily in the standard way which applies to image finite systems. Therefore *semidecidability of bisimilarity* is what matters here. In similar cases, the key point is to show that the bisimilarity case always has a *finite* (or finitely presented) *witness* whose validity can be checked algorithmically. Here, for the one-counter processes, the role of such witnesses is played by (descriptions of) semilinear sets; this approach was already used by Jančar (1993) and Esparza (1995). More precisely, the witnesses are presented here as certain 'regular' colourings of the plane which can be viewed as (special cases of) semilinear sets.

Roughly speaking, the existence of such witnesses (i.e. semilinear bisimulations) for one-counter processes can be anticipated from the intuition that two bisimilar processes must have the same 'distance' (minimum number of steps) to a 'bottom process' (configuration with zero in the counter) when such bottom processes matter at all; it can be guessed that the counter values of such processes have to be somehow 'linearly' related then. In general, the possibility of an algorithmic checking of a semilinear witness' validity can be derived from the decidability of Presburger arithmetic. Nevertheless a straightforward direct proof is provided in the special case here.

Another natural decidability question is that of *finiteness up to bisimulation* of a given process, which means its bisimulation equivalence with some finite-state process. This problem has been shown to be decidable for labelled Petri nets (Jančar and Esparza, 1996), which include BPP-processes. Burkart, Caucal, and Steffen (1996) showed the decidability for BPA-processes (where language regularity is well-known to be undecidable). The question for the whole class of pushdown processes is still open (while for the class of normed pushdown processes it is easily seen to be decidable). As an additional result, we demonstrate that finiteness up to bisimulation is also decidable for one-counter processes.

As mentioned above, one-counter processes can be 'almost' identified with labelled Petri nets with one unbounded place; but unlike Petri nets they can test for zero. Nevertheless the strategy used in the proof of decidability of finiteness up to bisimulation for labelled Petri nets (Jančar and Esparza, 1996) applies for them as well.

Section 2 states the main results precisely and contains two subsections with some known or easily derivable auxiliary results. The main decidability proof is given in Section 3 where the crucial technical part is separated in Subsection 3.1. Section 4 shows the decidability of finiteness, and Section 5 adds some further comments.

2 Basic Notions and Results

As usual, $\mathbb{N}$, $\mathbb{N}_+$ and $\mathbb{Z}$ will denote the sets of nonnegative, positive, and all integers respectively. A^* denotes the set of all finite sequences of elements of A; ε denotes the empty word. $A \setminus B$ denotes the set difference ($\{x \mid x \in A, x \notin B\}$). $\mathcal{P}(A)$ denotes the set of all subsets of A.

We begin by recalling some standard notions.

A *labelled transition system*, a *transition system* for short, is a pair $\mathcal{T} = (\mathcal{S}, \{\xrightarrow{a}\}_{a\in\mathcal{A}})$ where $\mathcal{S}$ is the set of *states*, $\mathcal{A}$ is the set of *actions* (or *action names*) and each $\xrightarrow{a}$ is a binary *(transition) relation* on $\mathcal{S}$ ($\xrightarrow{a} \subseteq \mathcal{S} \times \mathcal{S}$). By $E \to F$ ($E, F \in \mathcal{S}$) we mean that $E \xrightarrow{a} F$ for some a; $\to^*$ denotes the reflexive and transitive closure of the relation $\to$. By $E \to^* \mathcal{S}'$ ($\mathcal{S}'$ is *reachable from* E), where $\mathcal{S}' \subseteq \mathcal{S}$, we mean $E \to^* F$ for some $F \in \mathcal{S}'$. In the obvious sense, we also use $E \xrightarrow{u} F$ where $u \in \mathcal{A}^*$. By $|u|$ we denote the length of the sequence u.

A *transition system* $\mathcal{T} = (\mathcal{S}, \{\xrightarrow{a}\}_{a\in\mathcal{A}})$ is *finite* iff $\mathcal{S}$ and $\mathcal{A}$ are finite. $\mathcal{T}$ is *image finite* iff $succ(E) = \bigcup_{a\in\mathcal{A}} succ_a(E)$ is finite for any $E \in \mathcal{S}$, where we define $succ_a(E) = \{E' \mid E \xrightarrow{a} E'\}$.

In this paper, a *process* E is (being associated with) a state in a transition system which is clear from the context. When necessary, we denote the relevant transition system by $\mathcal{T}(E)$. Using the term of a *finite*, or rather a *finite-state*, *process* E, we mean that $\mathcal{T}(E)$ is finite;

similarly for an *image finite process.*

A binary relation $\mathcal{R}$ between processes is a *bisimulation* provided that whenever $(E, F) \in \mathcal{R}$, for each action a

$$\begin{array}{l}\text{if } E \xrightarrow{a} E' \text{ then there is } F' \text{ s.t. } F \xrightarrow{a} F' \text{ and } (E', F') \in \mathcal{R}, \text{ and}\\ \text{if } F \xrightarrow{a} F' \text{ then there is } E' \text{ s.t. } E \xrightarrow{a} E' \text{ and } (E', F') \in \mathcal{R}.\end{array}$$

Two processes E and F are *bisimulation equivalent*, or *bisimilar*, written $E \sim F$, if there is a bisimulation $\mathcal{R}$ relating them.

Notice that any union of bisimulations is a bisimulation, and that equivalence $\sim$ is the maximal bisimulation.

Now we define the one-counter processes (as a special case of the pushdown processes defined e.g. by Stirling, 1996).

Definition 2.1 *Suppose a given collection (i.e.* a one-counter machine*)* $M = (Q, \{X, Z\}, \mathcal{A}, \mathcal{B})$ *where* Q *is a finite set of* control states, X, Z *are* stack symbols, $\mathcal{A}$ *is a finite set of* actions, *and* $\mathcal{B}$ *is a finite set of* basic transitions, *each being either of the form* $pX \xrightarrow{a} q\alpha$, $\alpha \in \{\varepsilon, X, XX\}$, *or of the form* $pZ \xrightarrow{a} q\beta Z$, $\beta \in \{\varepsilon, X\}$, *where* $p, q \in Q$ *and* $a \in \mathcal{A}$. *The transition system* $\mathcal{T}_M$ *corresponding to* M *is stipulated to have states, called* one-counter processes, *of the form* $pXX \ldots XZ$*; for convenience, a process* pX^mZ *will be denoted by* $p(m)$.

The transition relations $\xrightarrow{a}$ $(a \in \mathcal{A})$ *are determined by the basic transitions together with the following* prefix rule*: if* $pX \xrightarrow{a} q\alpha$ *then* $p(m) \xrightarrow{a} q(m - 1 + |\alpha|)$ *for any* $m \geq 1$.

Remark. For technical convenience, we restricted the basic transitions so that each changes the counter by 1 at most. Nevertheless the proofs could be easily modified for the unrestricted case.

Our main aim here is to show

Theorem 2.2 *Bisimulation equivalence is decidable for one-counter processes.*

More precisely it means that there is an algorithm which inputs (descriptions of) two one-counter processes $p(m)$, $q(n)$ together with the respective one-counter machines M, M', and after a finite amount of time answers whether or not $p(m) \sim q(n)$.

In fact, we only consider the case $M = M'$ which can be easily achieved by defining a disjoint union of two machines in the natural way.

An additional result is expressed in the following theorem; recall that a (general) process E is called *finite up to bisimulation* iff there is a finite-state process p s.t. $E \sim p$.

Theorem 2.3 *Finiteness up to bisimulation is decidable for one-counter processes.*

We finish this section with two subsections which contain some auxiliary notions and some known or easily derivable results. The first subsection deals with results concerning bisimilarity while the second one with the results concerning 'plane colourings'.

2.1 Auxiliary results concerning bisimilarity

Let us recall some folklore results.

The family $\{\sim_n \mid n \geq 0\}$ (of relations between processes) is defined inductively:

1/ $E \sim_0 F$ for all processes E, F
2/ $E \sim_{n+1} F$ iff for each a

if $E \xrightarrow{a} E'$ then there is F' s.t. $F \xrightarrow{a} F'$ and $E' \sim_n F'$, and
if $F \xrightarrow{a} F'$ then there is E' s.t. $E \xrightarrow{a} E'$ and $E' \sim_n F'$.

Proposition 2.4 *For image finite processes, $E \sim F$ iff $\forall n \geq 0 : E \sim_n F$.*

Let us call $\mathcal{T} = (\mathcal{S}, \{\xrightarrow{a}\}_{a \in \mathcal{A}})$ an *admissible system* iff the state set $\mathcal{S}$ is finite or countably infinite (identified with a set of strings over a finite alphabet), the action set $\mathcal{A}$ is finite, $\mathcal{T}$ is image finite, and all the successor functions $succ_a : \mathcal{S} \longrightarrow 2^{\mathcal{S}}$ are effectively computable.

Proposition 2.5 *Considering only admissible transition systems, all the relations $E \sim_n F$ ($n \in \mathbb{N}$) are decidable. Therefore the problem $E \not\sim F$ is semidecidable.*

Since one-counter processes are obviously admissible, semidecidability of *nonbisimilarity* follows for them.

Definition 2.6 *Given a transition system $\mathcal{T} = (\mathcal{S}, \{\xrightarrow{a}\}_{a \in \mathcal{A}})$, we define the class of all* n-incompatible processes *as $\mathrm{INC}_n^{\mathcal{T}} = \{E \mid \forall F \in \mathcal{S} : E \not\sim_n F\}$.*

Remark. Here we let E vary over the class of all processes. Nevertheless we shall only be interested in the (non)reachability questions of $\mathrm{INC}_n^{\mathcal{T}}$ from some E', in the transition system $\mathcal{T}' = \mathcal{T}(E')$; then just the intersection of $\mathrm{INC}_n^{\mathcal{T}}$ with the state set of $\mathcal{T}'$ is of interest.

More specific variants of the following two propositions were used by Jančar and Moller (1995), and Jančar and Esparza (1996).

Proposition 2.7 *Consider the next two conditions*

1. *$E \sim F$,*
2. *$E \sim_n F$ and $E \not\rightarrow^* \mathrm{INC}_n^{\mathcal{T}(F)}$.*

Given any n, 1 implies 2. Given any n s.t. $\sim_{n-1}$ coincides with $\sim_n$ (and hence with $\sim$) on $\mathcal{T}(F)$, 2 implies 1.

Corollary 2.8 *Let A be a finite transition system with k states. Then $p \sim_{k-1} q$ iff $p \sim_k q$ (iff $p \sim q$) for any pair of states p, q. Hence for any process E and a state p of A, we have $E \sim p$ iff $E \sim_k p$ and $E \not\rightarrow^* \mathrm{INC}_k^A$.*

The next proposition is obvious; it claims that a necessary condition for bisimilarity of two processes is their equal 'distance' to any chosen $\sim_n$-class (for any n).

Definition 2.9 *The* distance *of a process E to F, denoted by* $\mathrm{dist}(E,F)$*, is the length of the shortest sequence u s.t. $E \xrightarrow{u} F$; if F is not reachable from E, we put* $\mathrm{dist}(E,F)=\infty$*. For a set $\mathcal{F}$ of processes, we define* $\mathrm{dist}(E,\mathcal{F}) = min\{\mathrm{dist}(E,F) \mid F \in \mathcal{F}\}$.

Proposition 2.10 *If $E \sim F$ then* $\mathrm{dist}(E,\mathcal{F}) = \mathrm{dist}(F,\mathcal{F})$ *for any equivalence class $\mathcal{F}$ of $\sim_n$ on the class of all processes.*

2.2 Auxiliary results concerning colourings

As mentioned in the Introduction, the role of finite witnesses for bisimilarity will be played by plane colourings which are regular in a certain way. Now we make these notions precise. We begin with geometrical notions; note that we only consider lines with rational slopes from $[0,\infty]$ (∞ meaning the vertical direction) and nonempty (hence infinite) intersections with $\mathbb{N}\times\mathbb{N}$.

Definition 2.11 *By a* plane *we mean the set* $\mathbb{N}\times\mathbb{N}$*; a* point *is an element of* $\mathbb{N}\times\mathbb{N}$*. An* area *is a subset of* $\mathbb{N}\times\mathbb{N}$.

A line *is given by one of its points and its nonnegative rational slope—including* 0 *and* ∞.

A belt *$B \subseteq \mathbb{N}\times\mathbb{N}$ (with slope α) is the area determined by a pair of parallel lines (with slope α)—as the set of all points between them, including those on them.*

A belt is horizontal *(*vertical*) iff its slope is 0 (∞); in addition, it is* initial *iff it contains the point* $(0,0)$.

Now we define notions concerning colourings.

Definition 2.12 *By a* colouring *(of the plane) with colours from a* finite *set C we mean a function $c : \mathbb{N}\times\mathbb{N} \to C$.*

A colouring is periodic *iff there is a* period *$\delta \in \mathbb{N}_+$ s.t. $c(i,j) = c(i+\delta,j) = c(i,j+\delta)$ for almost all (i.e. all but finitely many) $i,j \in \mathbb{N}$.*

A restriction of a colouring c to a domain $A \subseteq \mathbb{N}\times\mathbb{N}$, denoted by $c|_A$ is periodic *iff there is a periodic colouring c' s.t. $c'|_A = c|_A$.*

Informally, a (purely) periodic colouring is determined by a square ($\delta\times\delta$ points) which 'tiles' the plane; Fig. 1 illustrates an example (with two colours). We call periodic also a colouring which is (purely) periodic outside of a (sufficiently large) initial square.

Definition 2.13 *A plane* colouring *c is* regular *iff there are finitely many belts $B_1, B_2, \ldots, B_n$ s.t. all $c|_{B_1}, c|_{B_2}, \ldots, c|_{B_n}$ and $c|_A$, where $A = (\mathbb{N}\times\mathbb{N}) \setminus \bigcup_{i=1}^{n} B_i$, are periodic.*

In other words, a regular colouring arises from a periodic 'background' colouring by changing the colours in finitely many belts so that each belt colouring is periodic as well.

We observe the following simple facts about regular colourings; in a/ we use a convention that $k/0 = \infty$ for $k \in \mathbb{N}_+$ while $0/0$ is not defined.

Lemma 2.14

a/ For a belt B with slope α, $c|_B$ is periodic iff there is a 'period vector' $(\delta_1, \delta_2) \in \mathbb{N} \times \mathbb{N}$, s.t. $\delta_2/\delta_1 = \alpha$ and for all but finitely many $(i,j) \in B$, $c(i,j) = c(i+\delta_1, j+\delta_2)$.

b/ If $c|_{B_1}$, $c|_{B_2}$ are periodic for parallel belts B_1, B_2 then $c|_{B_1 \cup B_2}$ is periodic as well.

Claim a/ tells that a belt colouring (a colouring restricted to the belt) is periodic iff it is composed from equal segments, with a possible different 'beginning' segment, as depicted in Fig. 2.

For observing Claim b/, we notice that δ_1, δ_2 in a/ can be safely replaced by $k\delta_1, k\delta_2$ for any $k \in \mathbb{N}_+$. Therefore using a common multiple, it is easy to unify δ_1, δ_2 for both parallel belts B_1, B_2. Notice that it does not matter whether the belts intersect or not.

Now it should be clear that any regular colouring can be given in the form depicted in Fig. 3; a sufficiently large initial rectangle (containing all intersections of any two nonparallel belts) has to be chosen to this aim.

Suppose a fixed finite colour set C. Without going into technical details, it is obvious that any regular colouring $c : \mathbb{N} \times \mathbb{N} \to C$ can be naturally described by a finite string in a fixed finite alphabet; we can e.g. consider a 'natural' description of the scheme sketched in Fig 3. Such descriptions are effective, i.e. there is an algorithm which, given a description of c and a point (i,j), computes $c(i,j)$.

Also the fact that all regular colourings can be algorithmically generated is clear:

Proposition 2.15 *Given a fixed finite colour set C, the set of all (descriptions of) regular colourings is effectively enumerable.*

We will be interested in colourings where the colour set C is a cartesian product of simpler sets (in fact, it will be a power of the set $\{black, white\}$).

Definition 2.16 *For two colourings $c_1 : \mathbb{N} \times \mathbb{N} \to C_1$ and $c_2 : \mathbb{N} \times \mathbb{N} \to C_2$, we define their* product *as the colouring $c : \mathbb{N} \times \mathbb{N} \to C_1 \times C_2$ where $c(i,j) = (c_1(i,j), c_2(i,j))$ for all (i,j). The product $\prod_{i \in I} c_i$ of finitely many colourings (I being a finite index set) is defined as the natural generalization.*

The next proposition can be easily verified (by induction on $|I|$).

Proposition 2.17 *The colouring $\prod_{i \in I} c_i$ (I being finite) is periodic (regular) exactly when all c_i are periodic (regular).*

We also use the notion of a periodic set.

Definition 2.18 *A* set *$A \subseteq \mathbb{N} \times \mathbb{N}$ is* periodic *iff its 'characteristic' colouring $c : \mathbb{N} \times \mathbb{N} \to \{0,1\}$, where $c(i,j) = 1$ iff $(i,j) \in A$, is periodic. A* one-dimensional set *$A \subseteq \mathbb{N}$ is* periodic *when it arises as the restriction of a periodic set to the domain $\{(i,0) \mid i \in \mathbb{N}\}$.*

Later we refer to the following easily verifiable facts:

Lemma 2.19

a/ *Given $A \subseteq \mathbb{N}$, if there are $k, \delta \in \mathbb{N}$ s.t. $\forall m > k : m \in A \Rightarrow m+\delta \in A$ then A is periodic.*

b/ *If $A_1, A_2 \subseteq \mathbb{N}$ are periodic then $A_1 \times A_2$ is periodic outside an initial vertical and an initial horizontal belt.*

c/ *A finite union of periodic sets is a periodic set.*

We shall finish with a lemma telling that a belt whose border is coloured periodically necessarily contains two different subbelts with a 'bottom cut' (one being a 'shift' of the other) whose borders are coloured equally (cf. Fig. 4). First we make the notions precise; we shall also introduce the notion of a *cube* which will be useful later.

Definition 2.20 *A* b-subbelt of a belt B, *with the* vertical bottom $n \in \mathbb{N}$, *is the set* $\{(i,j) \in B \mid i \geq n\}$; *the b-subbelt with* horizontal bottom n *is* $\{(i,j) \in B \mid j \geq n\}$. *When B is horizontal (vertical), only the vertical (horizontal) bottom is allowed.*

Definition 2.21 *For $v \in \mathbb{Z}\times\mathbb{Z}$ we use notation $v=\langle (v)_1, (v)_2 \rangle$; addition (subtraction) is performed componentwise in $\mathbb{Z}\times\mathbb{Z}$.*

For $u \in \mathbb{N}\times\mathbb{N}$ we define $\mathrm{cube}(u) = (\ \{(u)_1-1, (u)_1, (u)_1+1\}\times\{(u)_2-1, (u)_2, (u)_2+1\}\ ) \cap (\mathbb{N}\times\mathbb{N})$. *By* $\mathrm{cubetype}(u)$ *we mean the set* $\{u'-u \mid u' \in \mathrm{cube}(u)\}$. *(There are four cube types).*

The set $\mathrm{bord}(A)=\{\ u \in (\mathbb{N}\times\mathbb{N}) \setminus A \mid \mathrm{cube}(u) \cap A \neq \emptyset\ \}$ *denotes the* (outside) border *of an* area A.

Definition 2.22 *Two* areas $A_1, A_2 \subseteq \mathbb{N}\times\mathbb{N}$ *are* shift-equivalent *iff there is a (unique) $v \in \mathbb{Z} \times \mathbb{Z}$ s.t. $\forall u \in A_1 : u + v \in A_2 \wedge$* $\mathrm{cubetype}(u) = \mathrm{cubetype}(u+v)$ *and $\forall u \in A_2 : u - v \in A_1 \wedge$* $\mathrm{cubetype}(u) = \mathrm{cubetype}(u-v)$.

In addition, given a colouring c, A_1, A_2 are c-isomorphic *iff $\forall u \in A_1 : c(u) = c(u+v)$.*

Lemma 2.23 *Suppose $c|_{\mathrm{bord}(B)}$ is periodic for a belt B. Then B contains two different b-subbelts B_1, B_2 s.t.* $\mathrm{bord}(B_1)$ *and* $\mathrm{bord}(B_2)$ *are c-isomorphic.*

We just sketch the proof idea informally.

As the border of a belt B consists of two belts (or one belt) parallel to B, we can take a common period vector (δ_1, δ_2) due to Lemma 2.14, and examine b-subbelts with horizontal bottoms at $k+\delta_2, k+2\delta_2, k+3\delta_2, \ldots$ for sufficiently large k (if $\delta_2 = 0$, we use vertical bottoms and δ_1 instead). Since all the infinitely many mentioned bottoms have the same finite size, the pigeonhole principle guarantees that there are two of them which are coloured 'equally'.

3 Decidability of Bisimilarity

In this section we provide a proof of Theorem 2.2.

Recalling Proposition 2.5, it suffices to establish *semidecidability* of the question whether $p(m) \sim q(n)$.

As mentioned in the Introduction, we shall use a special instance of a general strategy to establish semidecidability of $E_1 \sim E_2$ for some class of processes.

The strategy can be described as follows:

> Input processes E_1, E_2
> **for** $i := 1, 2, 3, \ldots$ **do**
>
> > generate the 'candidate' $\mathcal{R}_i$;
> > check whether $\mathcal{R}_i$ is a *witness* for E_1, E_2, i.e. whether it represents a bisimulation relating E_1, E_2;
> > **if** it is the case **then** HALT

For applicability and correctness of the strategy, the next conditions have to be satisfied:

- S1. the $\mathcal{R}_i$'s are finite objects (representing possibly infinite sets) and the set $\{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \ldots\}$ is recursively enumerable,
- S2. there is an algorithm which, when given E_1, E_2 and an $\mathcal{R}_i$, decides whether $\mathcal{R}_i$ is a witness for E_1, E_2,
- S3. whenever $E_1 \sim E_2$, there is an $\mathcal{R}_i$ being a witness for them.

In our case, the bisimulation candidates will be the (descriptions of) *regular colourings* c related to the considered one-counter machine M.

Definition 3.1 *Given a one-counter machine M with control state set Q, by a* colouring c *(*related to M*) we mean the product $c = \prod_{p,q \in Q} c_{\langle p,q \rangle}$ where $c_{\langle p,q \rangle} : \mathbb{N} \times \mathbb{N} \to \{black, white\}$.*

Such a colouring c represents *the* relation $\mathcal{R}_c$ *on the state set of $\mathcal{T}_M$ defined as follows: $p(m)\ \mathcal{R}_c\ q(n)$ iff $c_{\langle p,q \rangle}(m,n) = black$.*

We have to verify conditions S1, S2, S3.

Condition S1 is obvious (cf. Prop. 2.15).

To establish condition S2, we first recall the effectiveness of the descriptions mentioned before Prop. 2.15; it implies an algorithm deciding whether or not $p(m)\ \mathcal{R}_c\ q(n)$.

For showing that we can check algorithmically if $\mathcal{R}_c$ is a bisimulation, we introduce the following notions.

Definition 3.2 *A* pattern with type *ct, for a cube type ct, is a function $(Q \times Q) \to \mathcal{P}(ct)$. The* pattern of *a* point u *in* colouring c*, denoted* $\mathrm{pat}(u,c)$*, is the function $(Q \times Q) \to \mathcal{P}(\mathrm{cubetype}(u))$ s.t. $\mathrm{pat}(u,c)(\langle p,q \rangle) = \{u' - u \mid u' \in \mathrm{cube}(u), c_{\langle p,q \rangle}(u') = black\}$.*

A point $u = (i,j)$ is acceptable *in c iff it 'does not violate bisimulation', i.e. for any p,q, where $c_{\langle p,q \rangle}(u) = black$, and any $p(i) \xrightarrow{a} p'(i')$ $(q(j) \xrightarrow{a} q'(j'))$ there is $q(j) \xrightarrow{a} q'(j')$ $(p(i) \xrightarrow{a} p'(i'))$ s.t. $c_{\langle p',q' \rangle}(i',j') = black$.*

Note that acceptability of u in c is determined by $\mathrm{pat}(u,c)$. Hence we can naturally define the set AP of all *acceptable patterns* (related to M). AP is surely effectively constructible, and the next lemma can be easily verified.

Lemma 3.3 *For any colouring c, $\mathcal{R}_c$ is a bisimulation iff $\forall u \in \mathbb{N}\times\mathbb{N} : \mathrm{pat}(u,c) \in AP$.*

The (non)appearance of given patterns in a regular colouring can be surely checked algorithmically; it suffices to explore a sufficiently large initial rectangle (which includes a smaller initial rectangle covering all intersections, and several periods of periodically coloured belts meeting a common multiple of all periods; recall Fig. 3).

We can state it as the next proposition, which finishes the verification of condition S2.

Proposition 3.4 *Given M, with the control state set Q, and a regular colouring $c = \prod_{p,q\in Q} c_{\langle p,q\rangle}$, it is decidable whether or not c represents a bisimulation ($\mathcal{R}_c$) between processes of M.*

Remark. As mentioned in the Introduction, regular colourings can be viewed as a special case of semilinear sets (in the sense of Ginsburg and Spanier, 1966). Using a 'bigger machinery', we could derive that checking whether a semilinear candidate is a bisimulation can be done by constructing the appropriate formula of Presburger arithmetic which is decidable (cf. e.g. Oppen, 1978).

It remains to show the validity of condition S3.

To the machine M, we define the (unique) colouring

$$c_M = \prod_{p,q\in Q} c^M_{\langle p,q\rangle}$$

by putting $c^M_{\langle p,q\rangle}(i,j) = \mathit{black}$ iff $p(i) \sim q(j)$ and $c^M_{\langle p,q\rangle}(i,j) = \mathit{white}$ iff $p(i) \not\sim q(j)$.

Condition S3 will be clear and the whole proof of Theorem 2.2 will be finished, once we show

Proposition 3.5 *For a one-counter machine M, the colouring c_M is regular.*

One step for demonstrating this is to show the next lemma.

Lemma 3.6 *Outside of finitely many belts, c_M is periodic. I.e. there are finitely many belts $B_1, B_2, \ldots, B_n$ s.t. $c_M|_A$, where $A = (\mathbb{N}\times\mathbb{N}) \setminus \bigcup_{i=1}^{n} B_i$, is periodic.*

By recalling Proposition 2.17 we can reformulate the previous lemma as follows.

Lemma 3.7 *Outside of finitely many belts, $c^M_{\langle p,q\rangle}$ is periodic for any pair of control states p, q.*

The technical proof of this lemma is left for Subsection 3.1.

Here we show the rest of the proof of Proposition 3.5. Informally speaking, it will follow from the fact that two shift-equivalent areas whose borders are coloured equally in c_M are coloured equally. It implies that a belt on a periodically coloured background is also coloured periodically (in c_M).

Proposition 3.8 *Let A_1, A_2 be two areas s.t.* $\mathrm{bord}(A_1)$ *and* $\mathrm{bord}(A_2)$ *are c_M-isomorphic. Then A_1, A_2 are c_M-isomorphic.*

Proof: Let $v = \langle (v)_1, (v)_2 \rangle$ be the shifting vector of A_1 to A_2.

By induction, it is easy to verify that for all $n \geq 0$ we have $\forall p, q : \forall (i,j) \in A_1 : p(i) \sim_n q(j) \Leftrightarrow p(i + (v)_1) \sim_n q(j + (v)_2)$. $\square$

Corollary 3.9 *c_M restricted to any belt with a periodic border is periodic.*

Proof: This can be easily derived from the previous proposition and Lemma 2.23. $\square$

Proposition 3.5 is an immediate consequence of Lemma 3.6, or equivalently Lemma 3.7, and Corollary 3.9.

3.1 Crucial technical proof

Here we show a proof for Lemma 3.7.

We need the notion of the underlying automaton A_M, related to a one-counter machine M, which behaves like M as long as the bottom of the stack is not reached; we also define the processes which are *basically incompatible* with (states of) A_M.

Definition 3.10 *Given a one-counter machine $M = (Q, \{X, Z\}, \mathcal{A}, \mathcal{B})$, the* underlying finite automaton A_M *is the transition system $(Q, \{\xrightarrow{a}\}_{a \in \mathcal{A}})$ where $p \xrightarrow{a} q$ iff some $pX \xrightarrow{a} q\alpha$ is in $\mathcal{B}$.*

Denoting $|Q| = k$, we define $\mathrm{BINC} = \{p(m) \mid p(m) \in \mathrm{INC}_k^{A_M}\} = \{p(m) \mid \forall q \in Q : p(m) \not\sim_k q$ *(q taken as a state in A_M)*$\}$.

Further we implicitly suppose a fixed one-counter machine M with control state set Q, $|Q| = k$; control states are denoted p, q, possibly with super- or subscripts.

When we observe that $p(m) \sim_k p$ for $m \geq k$, the next lemma is clear:

Lemma 3.11 BINC *is a finite, effectively computable, set. Moreover, $p(m) \in$* BINC *implies $m < k$.*

Due to corollary 2.8 we can add:

Lemma 3.12 *For $m \geq k$ (and any control state p), $p(m) \sim p$ iff $p(m) \not\rightarrow^*$* BINC.

We now explain the strategy of the proof for Lemma 3.7.

Recall that, for any p, q, $c^M_{\langle p,q \rangle}(i,j) = \mathit{black}$ iff $p(i) \sim q(j)$ and $c^M_{\langle p,q \rangle}(i,j) = \mathit{white}$ iff $p(i) \not\sim q(j)$. Our aim is to show that the black points are distributed periodically, outside of finitely many belts.

A necessary condition for $c^M_{\langle p,q \rangle}(i,j) = \mathit{black}$ is that $\mathrm{dist}(p(i), \mathrm{BINC}) = \mathrm{dist}(q(j), \mathrm{BINC})$ (cf. Proposition 2.10); therefore all points (i,j) s.t. $\mathrm{dist}(p(i), \mathrm{BINC}) \neq \mathrm{dist}(q(j), \mathrm{BINC})$ are white.

We shall show that all points (i,j) s.t. $\mathrm{dist}(p(i),\mathrm{BINC}) = \mathrm{dist}(q(j),\mathrm{BINC}) < \infty$ are lying in finitely many belts.

Thus only the points (i,j) s.t. $\mathrm{dist}(p(i),\mathrm{BINC}) = \mathrm{dist}(q(j),\mathrm{BINC}) = \infty$, i.e. $p(i) \not\to^* \mathrm{BINC}$, $q(j) \not\to^* \mathrm{BINC}$, will remain interesting.

Outside the vertical belt given by the pair of lines $((0,0),\infty)$, $((k-1,0),\infty)$ and the horizontal belt given by the pair of lines $((0,0),0)$,$((0,k-1),0)$, Lemma 3.12 guarantees that the colour of all such (interesting) points is black when $p \sim q$ (in A_M) and white when $p \not\sim q$. If it is white, there is no problem (the whole background is white). To deal with the black case ($p \sim q$), it suffices to show that such points are distributed periodically—outside an initial horizontal and an initial vertical belt. Recalling Lemma 2.19 b/, it suffices to show that, for any control state p, the (one-dimensional) set $\{m \mid p(m) \not\to^* \mathrm{BINC}\}$ is periodic; equivalently we can show that $\{m \mid p(m) \to^* \mathrm{BINC}\}$ is periodic.

The steps of the described strategy are now performed formally.

Notation. By $p(m) \to^*_{\geq r} q(n)$ ($r \in \mathbb{N}$) we mean that there is a path $p(m) = q_1(n_1) \to q_2(n_2) \to \ldots \to q_s(n_s) = q(n)$ s.t. $n_i \geq r$ for $i = 1, 2, \ldots, s$. By $p(m) \to^*_{\mathrm{POS}} q(n)$ (POSitive) we mean that $p(m) \to^*_{\geq 1} q(n)$.

Observe the obvious fact (used implicitly in what follows): if $r \geq 1$ then $p(m) \to^*_{\geq r} q(n)$ iff $p(m+\delta) \to^*_{\geq r+\delta} q(n+\delta)$ for any $\delta \in \mathbb{N}$. In particular $p(m) \to^*_{\mathrm{POS}} q(n)$ implies $p(m+\delta) \to^*_{\mathrm{POS}} q(n+\delta)$.

Lemma 3.13 *For any control state p (of the one-counter machine M), the set $\{m \mid p(m) \to^*$ $\mathrm{BINC}\}$ is periodic; therefore also $\{m \mid p(m) \not\to^* \mathrm{BINC}\}$ is periodic.*

Proof: Recall that we suppose M with the control state set Q, $|Q| = k$.

Given $p \in Q$, we have to show that $R = \{m \mid p(m) \to^* \mathrm{BINC}\}$ is periodic. For any $P \subseteq Q$ we define the set $R_P \subseteq R$ as follows: $m \in R_P$ iff there is a 'witness' path

$$p(m) = q_1(n_1) \to q_2(n_2) \to \ldots \to q_s(n_s) \in \mathrm{BINC} \qquad (1)$$

s.t. $q_i \in P$ for $i = 1, 2, \ldots, s'$ where $s' \leq s$ is the maximum number s.t. $n_i \geq 1$ for $i = 1, 2, \ldots, s'$ (the path has to go solely through the control states from P until the stack bottom is reached—if it happens at all).

It is clear that $R_Q = R$ and it suffices to show that all R_P are periodic. We proceed by induction on $|P|$.

When $P = \emptyset$ then R_P is obviously periodic ($R_P = \emptyset$ or $R_P = \{0\}$).

Now we show that R_P, $|P| > 0$, is periodic while supposing that each $R_{P'}$, $|P'| < |P|$, is periodic. Let some $m \geq 2k$ be in R_P (otherwise R_P is finite, hence periodic) and let (1) be a relevant witness path; recall that $n_s < k$ (Lemma 3.11). We can take the leftmost subsequence $q_{i_1}(m), q_{i_2}(m-1), \ldots, q_{i_{k+1}}(m-k)$; due to the pigeonhole principle, there is $q = q_{i_a} = q_{i_b}$ for $a \neq b$. Therefore $p(m) \to^*_{\geq n'_1} q(n'_1) \to^*_{\geq n'_2} q(n'_2) \to^* q_s(n_s) \in \mathrm{BINC}$ where $\delta = n'_1 - n'_2 > 0$, $n'_2 > 0$; hence $q(n+\delta) \to^*_{\geq n} q(n)$ for any $n > 0$.

We can write $R_P = R^q_P \cup R_{P \setminus \{q\}}$ where

$$R^q_P = \{m \in R_P \mid \text{there is a witness path with } q = q_i \text{ for some } i, 1 \leq i \leq s'\}.$$

Since $m \in R_P^q$ obviously implies $m + \delta \in R_P^q$, R_P^q is periodic (cf. Lemma 2.19 a/); $R_{P \setminus \{q\}}$ is periodic due to the induction hypothesis. Therefore R_P is periodic as well (cf. Lemma 2.19 c/). □

Corollary 3.14 *There are constants $\ell, \Delta \in \mathbb{N}_+$ s.t. for any control state p of M and any $m > \ell$, the value $(m \bmod \Delta)$ determines whether or not $p(m) \to^* \mathrm{BINC}$.*

Proof: For any p, the appropriate ℓ_p and (period) Δ_p exist due to Lemma 3.13. The constant ℓ desired here can be taken as the maximum of ℓ_p's and Δ can be taken as the product of Δ_p's. □

Notation. $\mathrm{dist}(p(m), \mathrm{BINC})$ will be denoted by $\mathrm{dist}(p(m))$ for short.

Now we show that $\mathrm{dist}(p(m))$ is, in fact, linear (when finite) in m with the provision that the coefficient depends on $m \bmod \Delta$. Here and further, Δ is taken from Corollary 3.14.

Lemma 3.15 *There is a constant $d \in \mathbb{N}$ and there are rational constants $k_{p,i}$ defined for all control states p and all $i \in \{0, 1, \ldots, \Delta - 1\}$ s.t. the following property holds:*

$\forall p, \forall m \in \mathbb{N} : \mathrm{dist}(p(m)) < \infty \Longrightarrow k_{p,j}m - d \leq \mathrm{dist}(p(m)) \leq k_{p,j}m + d$, where $j = m \bmod \Delta$.

Proof: Let us fix some p and i, $0 \leq i \leq \Delta - 1$. In the proof, we let m, m_0 range only over the set $\{x \in \mathbb{N} \mid x \bmod \Delta = i\}$.

We show that there are $k_{p,i}$ and $d_{p,i}$ s.t. $k_{p,i}m - d_{p,i} \leq \mathrm{dist}(p(m)) \leq k_{p,i}m + d_{p,i}$, by which we will be done (the desired d can be taken as the maximum of the set $\{d_{p,i} \mid p \in Q, 0 \leq i \leq \Delta - 1\}$).

Given $m \geq 2k$, $p(m) \to^* \mathrm{BINC}$ implies a *decreasing cycle*:

$$p(m) \to^*_{\mathrm{POS}} q(n + \delta) \to^*_{\geq n} q(n) \to^* \mathrm{BINC} \text{ for some } q, n > 0, \delta > 0.$$

Let $P = \{\, q \mid p(m) \to^*_{\mathrm{POS}} q(n) \to^* \mathrm{BINC}$ for some $m, n \,\}$. Suppose $q' \in P$ allows a decreasing cycle $q'(n + \delta_w) \xrightarrow{w}_{\geq n} q'(n)$ (for some action sequence w, $\delta_w > 0$, and all $n \geq 1$) with the *best decreasing rate*—i.e. $\delta_w / |w|$ is maximal. The existence of such q' can be easily derived (consider just simple cycles where no control state is repeated). Moreover we can safely suppose that δ_w is a multiple of Δ (otherwise we take w^Δ which yields the same decreasing rate), and thus $q'(n + \delta_w) \to^* \mathrm{BINC}$ iff $q'(n) \to^* \mathrm{BINC}$ for $n > \ell$, ℓ taken from Corollary 3.14.

Let us choose $m_0 > \ell + \delta_w + k$ s.t. $p(m_0) \xrightarrow{u}_{\mathrm{POS}} q'(n_0) \to^* \mathrm{BINC}$ for some u and $n_0, \ell < n_0 \leq \ell + \delta_w$; denote $\delta_u = m_0 - n_0$. Note that $p(m_0 + j\Delta) \xrightarrow{u}_{\mathrm{POS}} q'(n_0 + j\Delta) \to^* \mathrm{BINC}$ for any $j \geq 0$.

Now let $d_0 = |u|$, $d_1 = max\{\mathrm{dist}(p'(\ell + x)) \mid p' \in Q, x \in \{0, 1, \ldots, \delta_w\}$ and $\mathrm{dist}(p'(\ell + x))$ is finite $\}$. Then it is clear that for any $m > \ell + \delta_w + k$

$$\mathrm{dist}(p(m)) \leq d_0 + \Big((m - \delta_u - \ell)/\delta_w\Big)|w| + d_1$$

On the other hand it is easily verifiable that

$$\mathrm{dist}(p(m)) \geq \Big((m - \delta_u - \ell)/\delta_w - 1\Big)|w|.$$

Calculating the desired $k_{p,i}$, $d_{p,i}$ is now a technical routine ($d_{p,i}$ has to be chosen large enough so that it also 'covers' the finitely many m, $m \leq \ell + \delta_w + k$). $\Box$

Corollary 3.16 *There is a constant $d \in \mathbb{N}$ and there are rational constants $k_{(p,q,i,j)}$ defined for all control states p, q and all $i, j \in \{0, 1, \ldots, \Delta - 1\}$ s.t. the following property holds:*
$\forall p, q, \forall m, n \in \mathbb{N} : \mathrm{dist}(p(m)) = \mathrm{dist}(q(n)) < \infty \Longrightarrow k_{(p,q,i',j')}m - d \leq n \leq k_{(p,q,i',j')}m + d$, where $i' = m \bmod \Delta$, $j' = n \bmod \Delta$.

Proof: Given p, q, and i, j, we know that there are constants k_1, k_2 and d' s.t. $k_1 m - d' \leq \mathrm{dist}(p(m)) \leq k_1 m + d'$ and $k_2 n - d' \leq \mathrm{dist}(q(n)) \leq k_2 n + d'$. Then $\mathrm{dist}(p(m)) = \mathrm{dist}(q(n))$ implies $k_2 n - d' \leq k_1 m + d'$ and $k_2 n + d' \geq k_1 m - d'$. Hence we have $mk_1/k_2 - 2d'/k_2 \leq n \leq mk_1/k_2 + 2d'/k_2$. $\Box$

Lemma 3.7 now follows from Corollary 3.16 and Lemma 3.13 when recalling the strategy described before 3.13.

4 Decidability of Finiteness

Here we provide a proof for Theorem 2.3.

Semidecidability of the problem whether $p(m)$ is finite up to bisimulation follows from Theorem 2.2; we can generate all finite state processes F, viewed as special cases of one-counter processes, and to check for each of them whether $p(m) \sim F$.

Semidecidability of the negative problem will immediately follow when we show that

$p(m)$ is *not* finite up to bisimulation

iff

there is a path

$$p(m) \to^* p'(m_1) \to^*_{\mathrm{POS}} p'(m_2) \to^*_{\mathrm{POS}} q'(n_1) \to^*_{\mathrm{POS}} q'(n_2) \to^* \mathrm{BINC}$$

where $m_1 < m_2$, $n_1 > n_2$.

The existence of such a path ensures for any $i \geq 0$ that

$$p(m) \to^* p'(m_2 + i(n_1 - n_2)(m_2 - m_1)) \to^* q'(n_1 + i(m_2 - m_1)(n_1 - n_2)) \to^* \mathrm{BINC}$$

which implies that there are reachable states with arbitrarily large (but finite) distances to BINC – and this obviously implies that infinitely many pairwise nonbisimilar states are reachable from $p(m)$. The opposite direction can also be easily established.

5 Further Comments

The example of a pushdown process used by Stirling (1996)

$$pX \xrightarrow{a} pXX, pX \xrightarrow{c} q\varepsilon, pX \xrightarrow{b} r\varepsilon, qX \xrightarrow{d} sX, sX \xrightarrow{d} q\varepsilon, rX \xrightarrow{d} r\varepsilon$$

can be easily transformed into a one-counter process with the isomorphic transition system. This process can serve as an example of a one-counter process which is not equivalent to a BPA-process, nor a BPP-process; when we add a state q_{fin} and a rule $pX \xrightarrow{f} q_{fin}$, we get a one-counter process which is not equivalent to any normed pushdown process.

The decidability result for bisimilarity can be related to the language equivalence decidability by Valiant and Paterson (1975). In a certain sense the result here is more general—considering *nondeterministic* machines, nevertheless it does not subsume the mentioned result completely—not considering ε-transitions.

Abdulla and Čerāns (1998) provide a (long, technical) proof of decidability of simulation equivalence for one-counter nets (a subclass of one-counter processes, with no zero-tests).

A step to find a unified approach to all of the mentioned decidability results has been done by Jančar and Moller (1998), who give an alternative (and much shorter) proof for the result of Abdulla and Čerāns.

Examination of the proof here would reveal that the slope and thickness of the belts as well as the periods of the background can be calculated. Analysing this, we could find an upper complexity bound (several exponentials seemingly). Nevertheless, such an analysis is left for future (it might be possible within the mentioned unified approach).

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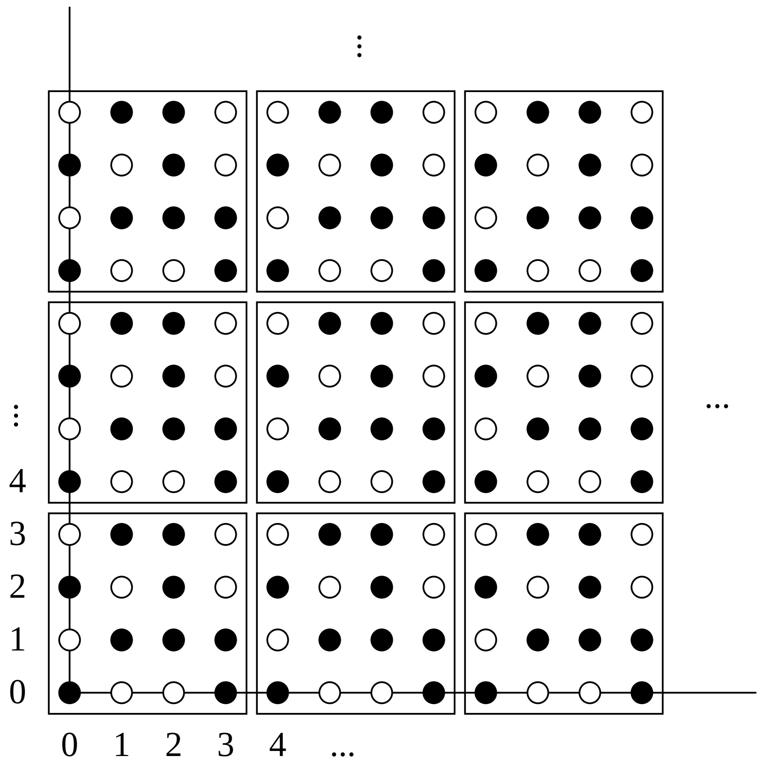

Figure 1: A periodic colouring

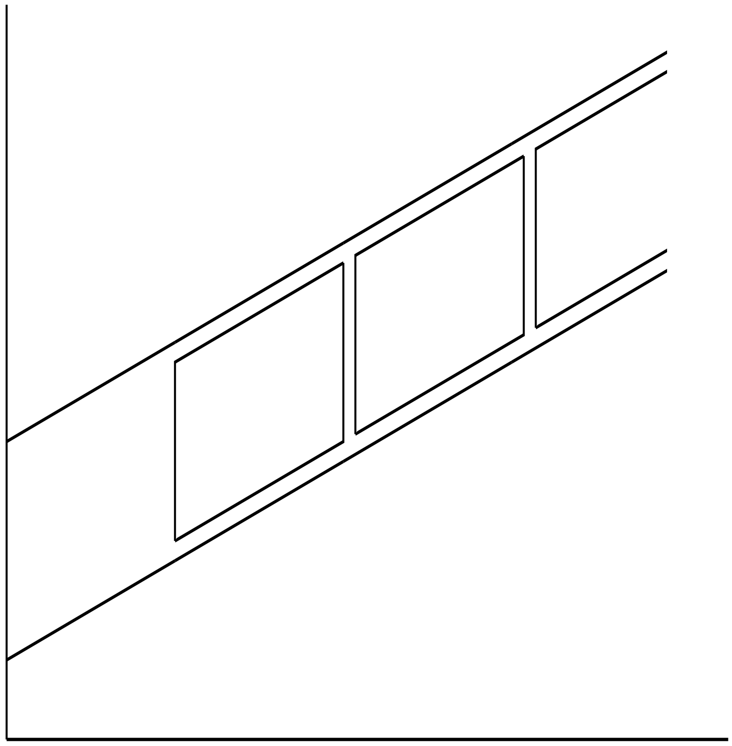

Figure 2: A periodic belt colouring

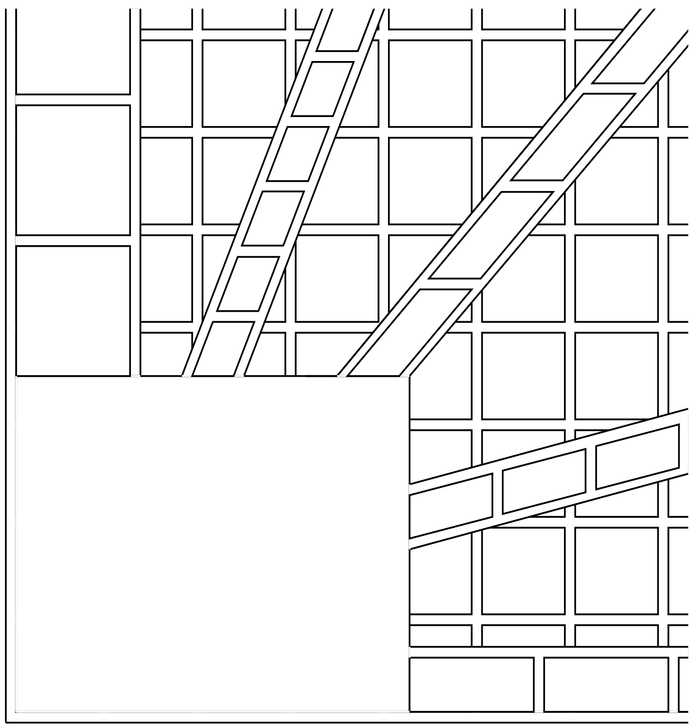

Figure 3: Scheme of a regular colouring

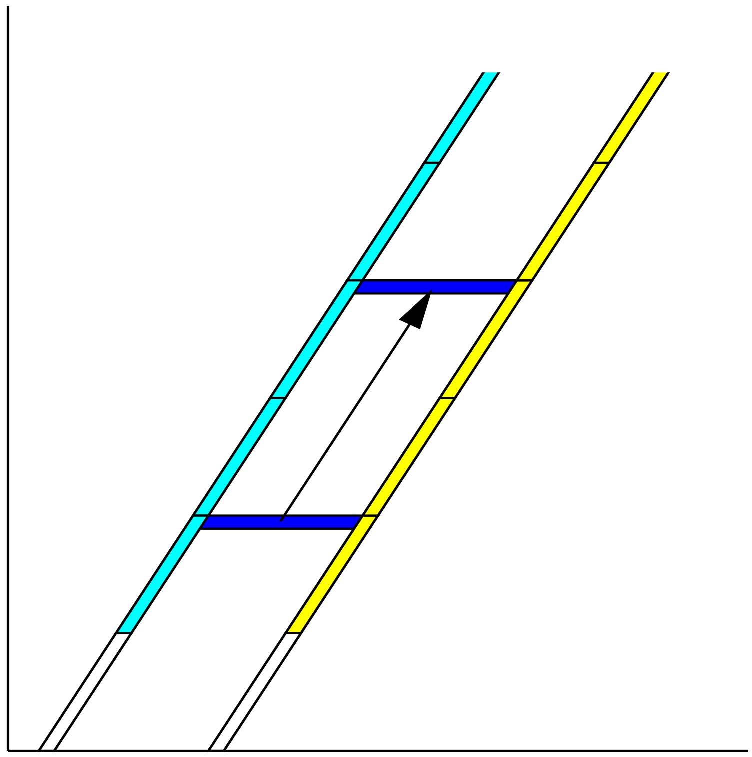

Figure 4: Two b-subbelts with equally coloured borders