A concise proof of Commoner's theorem

Petr Jančar
Univ. of Ostrava (jancar@osu.cz)
Bráfova 7, 701 03 Ostrava, Czech Republic

The accessible proofs of the well-known theorem characterizing liveness in free-choice nets are given in context of technical notions and lemmas, and thus seem a bit long (cf. [1,2]). Here a concise proof avoiding the mentioned technicalities is given.

The proof follows after a brief introduction of notation and definitions (for more details see e.g. [1,2]); the proof is intentionally concise, nevertheless all arguments should be easily verifiable from definitions.

A *net* N consists of the set S_N of *places*, the set T_N of *transitions*, and the relation $F_N \subseteq (S_N \times T_N) \cup (T_N \times S_N)$. For $x \in S_N \cup T_N$, $X \subseteq S_N \cup T_N$, we denote ${}^\bullet x = \{y \mid (y,x) \in F_N\}$, ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$; similarly for $x^\bullet$, $X^\bullet$. N is a *free-choice net* iff $p_1, p_2 \in {}^\bullet t \Longrightarrow p_1^\bullet = p_2^\bullet$ (for any $p_1, p_2 \in S_N$, $t \in T_N$); we suppose in addition that $p^\bullet \neq \emptyset$ for all $p \in S_N$. $X \subseteq S_N$ is a *trap* (a *siphon*) iff $X^\bullet \subseteq {}^\bullet X$ (${}^\bullet X \subseteq X^\bullet$). $M[t\rangle$ ($M[T\rangle$) denotes that the transition t (some $t \in T$) is enabled in the marking M. $[M\rangle$ means the set of all markings reachable from M. A *transition* t is *dead* in M, denoted by $t \in D_M$, iff $(\forall M' \in [M\rangle) : \neg(M'[t\rangle)$; t is *live* in M, $t \in L_M$, iff $(\forall M' \in [M\rangle) : t \notin D_{M'}$. A *marked net* (N, M_0) is *live* iff $L_{M_0} = T_N$. By $M\downarrow_Q$ we mean the restriction of M to the domain Q; $\mathbf{0}$ denotes the zero mapping (with the appropriate domain).

Theorem. (Commoner) *Given a free-choice net N, the next conditions are equivalent for any M_0: a/ (N, M_0) is live, and b/ every nonempty siphon of N contains a trap Q s.t. $M_0\downarrow_Q \neq \mathbf{0}$.*

non-a/$\Longrightarrow$ non-b/: If $L_{M_0} \neq T_N$ then there is $M \in [M_0\rangle$ s.t. $D_M \cup L_M = T_N$ and $D_M \neq \emptyset$ (cf. definitions of D_M, L_M). For any $t \in D_M$ there is $p_t \in {}^\bullet t$ s.t. $M(p) = 0$ and $p \notin (L_M)^\bullet$ (cf. free-choice). Hence $S = \{p_t \mid t \in D_M\}$ is a nonempty siphon, and $M\downarrow_S = \mathbf{0}$; therefore S can not contain a trap Q s.t. $M_0\downarrow_Q \neq \mathbf{0}$ ($M_0\downarrow_Q \neq \mathbf{0}$ implies $\forall M \in [M_0\rangle : M\downarrow_Q \neq \mathbf{0}$).

non-b/$\Longrightarrow$ non-a/ follows from the next lemma by noticing that there is no infinite decreasing chain in $<_P$: for an ordered subset $P = \{p_1, p_2, \ldots, p_n\}$ of S_N we put $M' <_P M$ iff $M'\downarrow_{\{p_1,p_2,\ldots,p_i\}} = M\downarrow_{\{p_1,p_2,\ldots,p_i\}}$ and $M'(p_{i+1}) < M(p_{i+1})$ for some i, $0 \leq i \leq n-1$.

Lemma. *Let us have a free-choice net N, $R \subseteq S_N$, and Q being the maximal trap (i.e. union of all traps) included in R. Then there is a particular ordering P of places in $R \setminus Q$ s.t. for any M, $M\downarrow_Q = \mathbf{0}$: if $M[R^\bullet\rangle$, or even if $R^\bullet \not\subseteq D_M$ in case R is a siphon, then there is $M' \in [M\rangle$ s.t. $M'\downarrow_Q = \mathbf{0}$ and $M' <_P M$.*

Proof of Lemma. By induction on $n = |R \setminus Q|$. The case $n = 0$ is trivial ($R^\bullet \subseteq D_M$ when R is a siphon). For $n > 0$, take $r \in R \setminus Q$ and $t_r \in r^\bullet$ s.t. $t_r^\bullet \cap R = \emptyset$ (R is not a trap !); then take $P = \{p_1, p_2, \ldots, p_n\}$ where $p_n = r$ and $P' = \{p_1, p_2, \ldots, p_{n-1}\}$ proves Lemma for $R \setminus \{r\}$. Suppose now M s.t. $M\downarrow_Q = \mathbf{0}$ and $M[R^\bullet\rangle$. Either $M[(R \setminus \{r\})^\bullet\rangle$ – then we can use the induction hypothesis ($M' <_{P'} M \Longrightarrow M' <_P M$) – or $M[r^\bullet\rangle$ – then $M[t_r\rangle$ (free choice !) and firing t_r does it. The case when R is a siphon follows since $R^\bullet \not\subseteq D_M$ implies $M'[R^\bullet\rangle$ for some $M' \in [M\rangle$, $M'\downarrow_R = M\downarrow_R$ (when ${}^\bullet R \subseteq R^\bullet$, the marking on R can not change until $R^\bullet$ becomes enabled).

[1] Desel J., Esparza J.: Free Choice Petri Nets; Cambridge Univ. Press, 1995
[2] Reisig W.: Petri Nets. EATCS Monographs on TCS, Vol. 4, Springer, 1985