A concise proof of Commoner’s theorem

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The accessible proofs of the well-known theorem characterizing liveness in free-choice nets are given in context of technical notions and lemmas, and thus seem a bit long (cf. [1,2]). Here a concise proof avoiding the mentioned technicalities is given.

The proof follows after a brief introduction of notation and definitions (for more details see e.g. [1,2]); the proof is intentionally concise, nevertheless all arguments should be easily verifiable from definitions.

A net $N$ consists of the set $S_N$ of places, the set $T_N$ of transitions, and the relation $F_N \subseteq (S_N \times T_N) \cup (T_N \times S_N)$. For $x \in S_N \cup T_N$, $X \subseteq S_N \cup T_N$, we denote $\mathbf{x} = \{ y \mid (y, x) \in F_N \}$, $\mathbf{X} = \bigcup_{x \in \mathbf{x}} \mathbf{x}$; similarly for $x^*, X^*$. $N$ is a free-choice net iff $p_1, p_2 \in \mathbf{x} \implies p_1^* = p_2^*$ (for any $p_1, p_2 \in S_N$). We suppose in addition that $\mathbf{p} \neq \emptyset$ for all $p \in S_N$. $X \subseteq S_N$ is a trap (a siphon) iff $X^* \subseteq \mathbf{X}$ ($\mathbf{X} \subseteq X^*$). $M(t) (M[T])$ denotes the transition $t$ (some $t \in T$) is enabled in the marking $M$. $[M]$ means the set of all markings reachable from $M$. A transition $t$ is dead in $M$, denoted by $t \in D_M$, iff $(\forall M' \in [M]) : \neg (M'[t])$; $t$ is live in $M$, $t \in L_M$, iff $(\forall M' \in [M]) : t \notin D_M$. A marked net $(N, M_0)$ is live iff $M_{L_0} = T_N$. By $\downarrow Q$ we mean the restriction of $M$ to the domain $Q$; $\emptyset$ denotes the zero mapping (with the appropriate domain).

**Theorem.** (Commoner) Given a free-choice net $N$, the next conditions are equivalent for any $M_0$: 

a/ $(N, M_0)$ is live, and b/ every nonempty siphon of $N$ contains a trap $Q$ s.t. $M_0 \downarrow Q \neq \emptyset$.

**non-a/implies non-b**: If $M_{L_0} \neq T_N$ then there is $M \in [M_0]$ s.t. $D_M \cup L_M = T_N$ and $D_M \neq \emptyset$ (cf. definitions of $D_M, L_M$). For any $t \in D_M$ there is $p \in \mathbf{x} t$ s.t. $M(p) = 0$ and $p \notin (L_M)^*$ (cf. free-choice). Hence $S = \{ p \mid t \in D_M \}$ is a nonempty siphon, and $M \downarrow S = \emptyset$; therefore $S$ cannot contain a trap $Q$ s.t. $M_0 \downarrow Q \neq \emptyset$ (hence for any $M \in [M_0] : M \downarrow Q \neq \emptyset$).

**non-b/implies non-a/** follows from the next lemma by noticing that there is no infinite decreasing chain in $\prec_P$: for an ordered subset $P = \{ p_1, p_2, \ldots, p_n \}$ of $S_N$ we put $M' \prec_P M$ iff $M' \downarrow (p_1, p_2, \ldots, p_i) = M \downarrow (p_1, p_2, \ldots, p_i)$ and $M'((p_{i+1}) < M((p_{i+1})$ for some $i$, $0 \leq i \leq n - 1$.

**Lemma.** Let us have a free-choice net $N$, $R \subseteq S_N$, and $Q$ being the maximal trap (i.e. union of all traps) included in $R$. Then there is a particular ordering $P$ of places in $R \setminus Q$ s.t. for any $M, M \downarrow Q = \emptyset$; if $M[R^*]$, or even if $R^* \not\subseteq D_M$, in case $R$ is a siphon, then there is $M' \in [M]$ s.t. $M' \downarrow Q = \emptyset$ and $M' \prec_P M$.

**Proof of Lemma.** By induction on $n = |R \setminus Q|$. The case $n = 0$ is trivial ($R^* \subseteq D_M$ when $R$ is a siphon). For $n > 0$, take $r \in R \setminus Q$ and $r' \in r^*$ s.t. $r^* \cap R = \emptyset$ ($R$ is not a trap!); then take $P = \{ p_1, p_2, \ldots, p_n \}$ where $p_n = r$ and $P' = \{ p_1, p_2, \ldots, p_{n-1} \}$ proves Lemma for $R \setminus \{ r \}$. Suppose now $M \succeq M[R^*]$. Either $M[(R \setminus \{ r \})^*] \prec_P M[R^*]$ then we can use the induction hypothesis ($M' \prec_P M \implies M' \prec_P M' \prec_M [R^*]$) - then we can use the induction hypothesis ($M' \prec_P M \implies M' \prec_P M'$) (free choice!) and firing $t_r$ does it. The case when $R$ is a siphon follows since $R^* \not\subseteq D_M$ implies $M'[R^*]$ for some $M' \in [M], M' \downarrow R = M \downarrow R$ (when $R^* \subseteq R^*$, the marking on $R$ can not change until $R^*$ becomes enabled).