Regularity of BPP is PSPACE-complete

Martin Kot

Department of Computer Science, FEI, VSB - Technical University of Ostrava,
17. listopadu 15, 708 33, Ostrava-Poruba, Czech Republic
martin.kot@vsb.cz

Abstract. Jančar has in 2003 shown that bisimilarity on Basic Parallel Processes (BPP) can be decided in polynomial space. But bisimilarity is studied also on various subclasses of BPP. We present summary of known complexity bounds of some such bisimilarity problems. Moreover is here shown that deciding regularity of a BPP is *PSPACE*-complete. This result was presented at winter school Movep'04 in PhD students section.

Keywords: basic parallel process, finite state system, bisimilarity, bisimulation equivalence, regularity, *PSPACE*-completeness.

1 Introduction

Equivalence checking is a well studied theoretical tool for a program verification. A program and its specification could be compared using some behavioral equivalence. One of the fundamental behavioral equivalences is bisimulation equivalence, also called bisimilarity. A program and a specification can be modeled using different models. We focus on well known model—basic parallel processes and some of its special forms. A BPP can model parallel composition of processes without synchronization. The superclass of BPP are Petri Nets which can model parallel composition with synchronization. BPP are useful because many problems are decidable for BPP and are not decidable for Petri Nets. For practical use of equivalence checking it is crucial not only to know if the problem is decidable but also to know how complex the decision procedure is.

A BPP is in general infinite-state model. It may be useful to know whether a particular BPP is really an infinite-state system or it has only a finite number of mutually nonbisimilar states and hence is bisimilar to some finite-state system. A problem of deciding existence of a finite-state system bisimilar to a given BPP is called *Regularity of BPP*. This problem was only known to be decidable and *PSPACE*-hard ([10]). We show an algorithm (published in [6]) working in *PSPACE* and hence combined with *PSPACE*-hardness we get *PSPACE*-completeness. We will not construct a bisimilar finite-state system because it may be in general of exponential size with respect to a given BPP.

In the section 2 we define basic notions. In the section 3 will be summary of some known results. In section 4 we show that deciding regularity of a BPP is *PSPACE*-complete.

2 Basic definitions and notation

A *labelled transition system* (LTS) is a triple $(S, \mathcal{A}, \longrightarrow)$, where S is a set of *states*, $\mathcal{A}$ is a finite set of *actions*, and $\longrightarrow \subseteq S \times \mathcal{A} \times S$ is a *transition relation*. We write $s \xrightarrow{a} s'$ instead of $(s, a, s') \in \longrightarrow$ and we extend this notation to elements of $\mathcal{A}^*$ in a natural way.

Let $(S, \mathcal{A}, \longrightarrow)$ be an LTS. A binary relation $\mathcal{R} \subseteq S \times S$ is a *bisimulation* iff for each $(s, t) \in \mathcal{R}$ and $a \in \mathcal{A}$ we have:

- $\forall s' \in S : s \xrightarrow{a} s' \Rightarrow (\exists t' : t \xrightarrow{a} t' \wedge (s', t') \in \mathcal{R})$, and
- $\forall t' \in S : t \xrightarrow{a} t' \Rightarrow (\exists s' : s \xrightarrow{a} s' \wedge (s', t') \in \mathcal{R})$.

States s and t are *bisimulation equivalent (bisimilar)*, written $s \sim t$, iff they are related by some bisimulation.

A BPP can be defined as a special form of Petri net called communication-free Petri net. Concretely a BPP is a tuple $(P, \mathit{Tr}, \text{PRE}, F, \lambda)$ where P is a finite set of *places*, Tr is a finite set of *transitions*, $\text{PRE} : \mathit{Tr} \to P$ is a function assigning an input place to every transition, $F : (\mathit{Tr} \times P) \to \mathbb{N}$ is a function assigning output places to each transition, and $\lambda : \mathit{Tr} \to \mathcal{A}$ is a labeling function. The set of output places of the transition t we will denote by $\text{SUC}(t) = \{p \mid F(t, p) > 0\}$.

Let $P = \{p_1, p_2, \ldots, p_k\}$ be a set of places. A *marking* is a function $M : P \to \mathbb{N}$ which assigns number of tokens to each place. We use $\mathcal{S}_\Sigma$ to denote the set of all markings.

A transition t is *enabled* in a marking M iff $M(\text{PRE}(t)) > 0$. Performing a transition, written $M \xrightarrow{t} M'$, means

$$M'(p) = \begin{cases} M(p) - 1 + F(t, p) & \text{if } p = \text{PRE}(t) \\ M(p) + F(t, p) & \text{otherwise} \end{cases}$$

A BPP is called *normed*, denoted nBPP, iff from each marking we can reach an empty marking (i.e., $M(p_i) = 0$ for each $p_i \in \mathcal{S}_\Sigma$) by performing a sequence of transitions.

An LTS $(S, \mathcal{A}, \longrightarrow)$ corresponds to a BPP where $S = \mathcal{S}_\Sigma$ and $M \xrightarrow{a} M'$ iff there is some $t \in \mathit{Tr}$ such that $\lambda(t) = a$ and $M \xrightarrow{t} M'$.

A set of places $R \subseteq P$ is a *trap* iff $\forall t : \text{PRE}(t) \in R \Rightarrow (\exists p \in R : F(t, p) \geq 1)$. A trap R is called *important* if $M \sim M'$ implies $M|_R = \mathbf{0} \Leftrightarrow M'|_R = \mathbf{0}$.

A *finite state system* (FS) is a LTS with finite set of states.

For a LTS $(S, \mathcal{A}, \longrightarrow)$ we define the distance function $\mathit{dist} : (S \times S) \to \mathbb{N}_\omega$ as $\mathit{dist}(s, s') = \min\{\mathit{length}(w) \mid w \in \mathcal{A}^*, s \xrightarrow{w} s'\}$. (We define $\min \emptyset = \omega$.)

The set of *DD-functions* introduced in [3] is defined inductively as follows:

- For each $a \in \mathcal{A}$ the function $dd_a : S \to \mathbb{N}_\omega$, defined as $dd_a(s) = \min\left\{ \mathit{dist}(s, s') \mid \neg\exists s'' : s' \xrightarrow{a} s'' \right\}$, is a DD-function.
- Let $\mathcal{F} = (d_1, \ldots, d_k)$ be a tuple of DD-functions. For $s, s' \in S$, such that $s \xrightarrow{a} s'$, we define the *change* $\delta = \mathcal{F}(s') - \mathcal{F}(s)$, which is a k-tuple of values from $\mathbb{N}_{\omega,-1}$. (We put $\omega - \omega = \omega$.) A function $dd_{(a,\mathcal{F},\delta)} : S \to \mathbb{N}_\omega$, defined as $dd_{(a,\mathcal{F},\delta)}(t) = \min\left\{ \mathit{dist}(t, t') \mid d_i(t') < \omega \text{ for all } i, \text{ and } \mathcal{F}(t'') - \mathcal{F}(t') \neq \delta \text{ for all } t' \xrightarrow{a} t'' \right\}$, is a DD-function.

Informally, the value of DD-function $d(s)$ represents 'distance' from s to the nearest state where certain kind of transitions is disabled, (and $d(s) = \omega$ if there is no such state reachable from s), in particular, dd_a is the distance to a state where transitions labelled with a are disabled.

All DD-functions are *bisimulation invariant*, i.e., if s and s' are bisimilar then $d(s) = d(s')$ for all DD-functions d. So equality of the values of all DD-functions is a necessary condition for two places being bisimilar. In the case of BPP (and all other finite-branching systems) this condition is also sufficient. Hence $s \sim s'$ iff $d(s) = d(s')$ for each DD-function d. In [3] was shown that, for any BPP, DD-functions coincide with so called 'norms':

Given $Q \subseteq \mathcal{S}_\Sigma$, we define function $\textsc{norm}_Q$ by

$$\textsc{norm}_Q(M) = \min \left\{ \, dist(M, M') \;\mid\; M'(p) = 0 \text{ for each } p \in Q \, \right\}.$$

Each $\textsc{norm}_Q$ is a linear function, i.e, for each $p \in P$ there is $c_p \in \mathbb{N}_\omega$ such that $\textsc{norm}_Q(M) = \sum_p c_p \cdot M(p)$

For all bisimulation invariant linear functions $L(M) = \sum_p c_p \cdot M(p)$ (and hence for all DD-functions) the set $R_L = \{p | c_p = \omega\}$ is an important trap.

We define relation $\preceq$ on the set of all markings $\mathcal{S}_\Sigma$ as follows. For markings $M = (x_1, x_2, \ldots, x_k)$ and $M' = (x'_1, x'_2, \ldots, x'_k)$ it holds $M \preceq M'$ iff $x_1 \leq x'_1 \wedge x_2 \leq x'_2 \wedge \ldots \wedge x_k \leq x'_k$. This relation is reflexive and transitive hence it is quasi-order. Obviously for every infinite sequence $M_1, M_2, \ldots$ of markings there exist $i < j \in \mathcal{N}$ such that $M_i \preceq M_j$ hence the relation is well-quasi-ordering. In an obvious manner is defined relation $\prec$.

3 Known results related to bisimilarity on BPP

Let us first define three known problems concerning bisimilarity on BPP.

Bisimilarity on BPP – Given BPPs Σ_1, Σ_2 together with initial markings M_{I_1}, M_{I_2}, is $M_{I_1} \sim M_{I_2}$?

Bisimilarity of BPP and FS – Given BPP Σ with initial marking M_I and FS Δ with initial state s_I, is $M_I \sim s_I$?

Regularity of BPP – Given BPP Σ together with initial marking M_I, does a FS with initial state s_I exist such that $M_I \sim s_I$?

Similar problems can be defined for normed BPP by replacing each BPP in instances by a nBPP.

The table 1 shows best currently known complexity bounds for our three problems on BPP and nBPP. It is partially obtained from [9] and updated.

Author of this paper cooperated on two most recent results [5] and [7].

4 The regularity of a BPP is *PSPACE*-complete

The regularity of BPP was known to be decidable and *PSPACE*-hard. Using some methods from [3] we show an algorithm running in polynomial space for this problem. Hence we get that regularity of BPP is *PSPACE*-complete.

	BPP	nBPP
Bisimilarity	$\in$ *PSPACE* [3] *PSPACE*-hard [10]	$\in P$ [2] ($O(n^3)$ [5]) P-hard [1]
Bisimilarity with FS	$\in P$ ($O(n^4)$) [7] P-hard [1]	$\in P$ [2] ($O(n^3)$ [5]) P-hard [1]
Regularity	decidable [4] *PSPACE*-hard [10]	$\in$ *NL* [8] *NL*-hard [10]

Table 1. Known complexity bounds of problems concerning bisimilarity on BPP

We have given a BPP $\Sigma = (P, Tr, \text{PRE}, F, \lambda)$ and initial marking M_I. The question is whether there is a FS Δ bisimilar with Σ. In the case of positive answer we will not construct existing Δ (this can be exponential to the size of Σ).

In [3] an algorithm is presented which given a BPP Σ constructs in polynomial space a mapping $\mathcal{C}_\Sigma$. Two markings M_1, M_2 of Σ are bisimilar iff $\mathcal{C}_\Sigma(M_1) = \mathcal{C}_\Sigma(M_2)$. Moreover $\mathcal{C}_\Sigma$ is n-tuple $(L_1, L_2, \ldots, L_n)$ of linear functions. Each L_i is a DD-function and in fact the norm of some set of places of Σ.

Lemma 1. *A BPP Σ is regular iff there is finite number of mutually nonbisimilar markings.*

Proof. '$\Leftarrow$' Let the number of mutually nonbisimilar markings be finite. We can define a LTS $\Delta = (S, A, \{\xrightarrow{a}\}_{a\in A})$ where $S = \{[M]_\sim \mid M$ is a marking of $\Sigma\}$ and transitions are defined in obvious manner — $[M_1]_\sim \xrightarrow{a} [M_2]_\sim$ if there are markings $M_1' \in [M_1]_\sim$, $M_2' \in [M_2]_\sim$ and a transition t such that $M_1' \xrightarrow{t} M_2', \lambda(t) = a$. Then Δ is a finite state system bisimilar with Σ. It follows that Σ is regular.

'$\Rightarrow$' Now let the number of mutually nonbisimilar markings be infinite. We can not construct a FS which has one state for each equivalence class on markings. If two markings are nonbisimilar they can not be both bisimilar with the same state of finite state system. Hence there is not any FS bisimilar with Σ and Σ is not regular. $\square$

Because $\mathcal{C}_\Sigma(M_1) = \mathcal{C}_\Sigma(M_2)$ for $M_1 \sim M_2$ system is regular iff we have a finite number of different possible values of $\mathcal{C}_\Sigma$ on reachable markings of Σ. An infinite number of values of $\mathcal{C}_\Sigma$ is possible iff at least one of the functions L_i has an infinite number of the possible values.

Lemma 2. *Norm function L has infinite number of different values on markings of Σ iff in the BPP Σ, there is a subnet of the form as on the figure 1 (in Σ places and transitions can have more output arcs and places also input arcs) such that:*

1. $n \geq 1$ *(in the case that $n = 1$, $F(t_1, p_1) = 1$)*
2. $p_i \neq p_j$ *for* $i \neq j$, $1 \leq i, j \leq n$

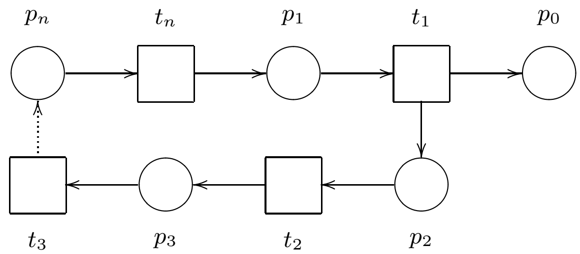

Fig. 1. Cycle possibly causing unregularity of a BPP

3. $(\cup_{i=1}^{n} \textsc{suc}(t_i)) \cap R_L = \emptyset$
4. *It is possible that* $p_0 = p_i$ *for some* $1 \leq i \leq n$ *but also that* $p_0 \neq p_i$ *for all* $1 \leq i \leq n$
5. $0 < c_{p_0} < \omega$ *or there is a sequence of transitions which takes token from* p_0*, gives token to some place* p *(*$0 < c_p < \omega$*) and does not mark* R_L
6. *there is a marking* M *reachable from initial marking* M_I *such that* $L(M) < \omega$ *and* $M(p_i) > 0$ *for some* $1 \leq i \leq n$

Proof. '$\Rightarrow$' We suppose a BPP Σ such that there is an infinite number of different values of function L on reachable markings. L is a linear function, i.e, for each $p \in P$ there is $c_p \in \mathbb{N}_\omega$ such that $L(M) = \sum_p c_p \cdot M(p)$.

The BPP is finitely branching hence in the unfolding of Σ must be at least one infinite sequence of markings $M_1, M_2, \ldots$ such that the sequence $L(M_1), L(M_2), \ldots$ contains infinitely many different values. Now we can take an infinite subsequence $M_1', M_2', \ldots$ such that $L(M_1'), L(M_2'), \ldots$ is strictly growing. The relation $\preceq$ is well-quasi-ordering hence there are M_i', M_j' in our sequence such that $i < j$, $M_i' \preceq M_j'$. From the fact that $L(M_i') < L(M_j')$ and that L is linear function follows $M_i' \prec M_j'$. The sequence of transitions leading from M_i' to M_j' could be repeated infinitely often which corresponds to the transitions $t_1, \ldots, t_n$ on the figure 1. Moreover each repetition have to generate at least one new token hence there is the arc from t_1 to p_0 on the figure.

From the fact that $L(M_1), L(M_2), \ldots$ contains infinitely many different values follows, that the trap R_L is not marked in any marking $M_1, M_2, \ldots$. $L(M_i') < L(M_j')$ hence it have to be possible to find p_0 such that $0 < c_{p_0} < \omega$ or transport token form p_0 to p $(0 < c_p < \omega)$ without marking R_L.

'$\Leftarrow$' If the structure depicted on figure 1 exist in the BPP and all conditions from our lemma are satisfied then then we can reach (without marking a trap R_L) a marking from which it is possible to repeat the sequence of transitions $t_1, t_2, \ldots, t_n$ infinitely many times. Moreover the value of L stays finite and is growing. This means that L can reach infinitely many different values. $\square$

Theorem 1. *The regularity of a BPP is in PSPACE and hence is PSPACE-complete.*

Proof. A BPP has a finite number of places and transitions. We can check all possible subsets of places if they correspond the structure depicted on figure 1. This can be obviously done in polynomial space. If such a structure is found we can check for each DD-function L the conditions from lemma 2. This can be done in polynomial space using algorithms from [3] for computation of all DD-functions and important traps. If a cycle fulfilling all conditions is found, the BPP system is not regular. In the other case the system is regular. □

5 Future work

As the table 1 together with section 4 suggests there are known quite proper complexity bounds for all problems concerning bisimilarity on BPP and its subclasses. But it is worth to study also bisimilarity between two different models. Recent time we are working on bisimilarity algorithm deciding whether a given BPP is bisimilar to a given Basic Process Algebra (BPA). BPA can model only sequential composition. We also try to describe a class of infinite-state systems which can be modeled using the parallel composition only as well as using the sequential composition only. Some preliminary results will be presented on Express'05 - an affiliated workshop of Concur'05.

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