# Tutorial 5

**Exercise 1:** Consider problem

NAME: ILP (*Integer Linear Programming*)

INPUT: Matrix $A$ of dimension $m \times n$ and column vector $b$ of size $m$ whose elements are integers.

QUESTION: Is there an integer column vector $x$ (of size $n$) such that $Ax \leq b$?

At first, show some small (but not completely trivial) instance of ILP, for which that answer is YES, and some instance, for which the answer is NO.

Explain in detail what should be done to show that there is a polynomial reduction from 3-SAT to ILP.

What can be said about complexity of ILP when such polynomial reduction is presented?

Try to show such reduction.

Try to think about whether ILP belongs to NP.

This is indeed the case. But ILP is an example of a problem where membership in NP is not obvious, and its proof is nontrivial. This is the difference compared with problems in NP discussed before.

(We will just mention here without a proof, that in fact, if there exists an integer solution of inequality $Ax \leq b$, then there exists a solution, which is "small enough" – it can be represented by number of bits, which is polynomial with respect to number of bits necessary for representation of $A$ and $b$; so such solution can be guessed and verified in polynomial time.)

**Exercise 2:** Show that if $\mathsf{P} = \mathsf{NP}$, then every problem from $\mathsf{P}$, with the exception of trivial problems where the answer is always YES or always NO, is $\mathsf{NP}$-complete.

**Exercise 3:** Show that if SAT problem could be solved with an algorithm with time complexity $\mathcal{O}(f(n))$, where $f(n)$ is a polynom, then also the following problem could be solved in polynomial time:

INPUT: Formula $\varphi$ of propositional logic.

OUTPUT: Truth valuation $v$, for which the formula $\varphi$ is true, or information that there is no such valuation.

What would be time complexity of the algorithm that would solve this problem?

**Exercise 4:** Show that the following problem is $\mathsf{NP}$-complete:

INPUT: Formula $\varphi$ of propositional logic.

QUESTION: Are there at least two truth valuations, for which the formula $\varphi$ is true?

**Exercise 5:** Show that the following problem is NP-complete:

> INPUT: Propositional logic formula $\varphi$ in conjunctive normal form.
>
> QUESTION: Is there a truth valuation $\nu$, for which every clause of the formula $\varphi$ contains at least one literal that has value $1$ in valuation $\nu$, and at least one literal that has value $0$ in valuation $\nu$?

*Hint:* Consider a reduction from 3-SAT problem where every clause

$$(\ell_1 \vee \ell_2 \vee \ell_3)$$

is replaced with a pair of clauses

$$(\ell_1 \vee \ell_2 \vee y_i) \qquad \text{a} \qquad (\neg y_i \vee \ell_3 \vee b)$$

where $y_i$ is a new variable for each clause $C_i$, and $b$ is a new variable common for whole formula.

Justify the correctness of this reduction.

**Exercise 6:** Show that 2-SAT problem is in PTIME:

> INPUT: A proposition logic formula $\varphi$ in conjunctive normal form where every clause contains at most 2 literals.
>
> QUESTION: Is the formula $\varphi$ satisfiable?

**Exercise 7:** Show that the variant of SAT problem where input instances are restricted to so called Horn formulas, can be solved in polynomial time.

Horn formulas are formulas of propositional logic that are in conjunctive normal form where moreover each clause contains at most one positive literal.

So the goal is to show that the following problem is in PTIME:

> INPUT: A Horn formula $\varphi$.
>
> QUESTION: Is the formula $\varphi$ satisfiable?