Tutorial 7

Exercise 1: Show that the following problems can be solved with a deterministic algorithm with a logarithmic space complexity (i.e., with space complexity $\mathcal{O}(\log n)$ where n is the size of an input).

Remark: You can assume that all numbers in inputs and outputs are represented in binary.

a) INPUT: A pair of natural numbers x and y.
OUTPUT: The value of the sum $x + y$.

b) INPUT: A sequence of natural numbers $x_1, x_2, \dots, x_k$.
OUTPUT: The value of the sum $x_1 + x_2 + \cdots + x_k$.

Remark: As the size of the input consider the total number of bits necessary to represent the whole sequence $x_1, x_2, \dots, x_k$.

c) INPUT: A pair of natural numbers x and y.
OUTPUT: The value of the product $x \cdot y$.

d) INPUT: A word w containing different types of parenthesis ($[_1$, $]_1$, $[_2$, $]_2$, …, $[_r$, $]_r$).
QUESTION: Is w correctly parenthesised?

Remark: A sequence is correctly parenthesised if it is generated by the following grammar:

$$A \longrightarrow \varepsilon \mid AA \mid [_1 A\,]_1 \mid [_2 A\,]_2 \mid \cdots \mid [_r A\,]_r$$

Exercise 2: Show that the following problems are NL-complete:

a) INPUT: A nondeterministic finite automaton $\mathcal{A}$ and a word w.
QUESTION: Does the automaton $\mathcal{A}$ accept the word w (i.e., is $w \in \mathcal{L}(\mathcal{A})$)?

b) INPUT: A deterministic finite automaton $\mathcal{A}$.
QUESTION: Is $\mathcal{L}(\mathcal{A}) = \emptyset$?

c) INPUT: A deterministic finite automaton $\mathcal{A}$.
QUESTION: Is $\mathcal{L}(\mathcal{A}) = \Sigma^*$?

d) INPUT: Deterministic finite automata $\mathcal{A}_1$ and $\mathcal{A}_2$.
QUESTION: Is $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$?

e) INPUT: A directed graph G.
QUESTION: Is the graph G strongly connected?

Remark: A graph is strongly connected if for each pair of its nodes u and v there exists a path from u to v.

f) Input: A finite set X, an associative binary operation $\circ$ on the set X (represented as a table specifying the value $x \circ y$ for each pair $x, y \in X$), a subset $S \subseteq X$, and an element $t \in X$.

Question: Can the element t be generated from elements of the set S?

Remark: An element t can be generated from elements of the set S if there exists a sequence $x_1, x_2, \ldots, x_k$ of elements of the set S such that

$$t = x_1 \circ x_2 \circ \cdots \circ x_k$$

Exercise 3: Show that the following problems are P-complete.

Hint: P-hardness of these problems can be shown for example by describing a logspace reduction from Monotone Circuit Value Problem (MCVP).

a) Input: A combinatorial game played by two players whose graph is given explicitly, i.e., all positions and possible moves are listed explicitly. For each position, it is specified, which of players has the following term. An initial position α is specified.

Question: Does *Player I* have a winning stratedy in the game starting in position α?

b) Input: A context-free grammar $\mathcal{G}$ and a word $w \in \Sigma^*$.

Question: Does the word w belong to the language generated by the grammar $\mathcal{G}$ (i.e., is $w \in \mathcal{L}(\mathcal{G})$)?

c) Input: A context-free grammar $\mathcal{G}$.

Question: Is $\mathcal{L}(\mathcal{G}) = \emptyset$?

d) Input: A context-free grammar $\mathcal{G}$.

Question: Is the language $\mathcal{L}(\mathcal{G})$ infinite?

e) Input: A finite set X, a binary operation $\circ$ on the set X (given as a table specifying the value $x \circ y$ for each pair $x, y \in X$), a subset $S \subseteq X$, and an element $t \in X$.

Question: Can the element t be generated from the elements of the set S?

Remark: An element t can be generated from elements of a set S if there exists an expression consisting of constants representing elements of the set S, on which the operation $\circ$ is applied in an arbitrary way, such that the value of this expression is t.

Other way, how to say the same thing is to say that t belongs to the smallest set Y (where $Y \subseteq X$) that satisfies the following two conditions:

- $S \subseteq Y$
- for each pair of elements $x, y \in Y$ it holds that $x \circ y \in Y$.