Introduction to Theoretical Computer Science

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On these pages you will find:

- Information about the course
- Study texts
- Slides from lectures
- Exercises for tutorials
- Recent news for the course
- A link to a page with animations

Requirements

- **Credit** (22 points):
  - Written test (22 points) — it will be written on a tutorial

  The minimal requirement for obtaining the credit is 7 points.

  A correcting test for 14 points.

- **Exam** (78 points)
  - A written exam consisting of three parts (26 points for each part); it is necessary to obtain at least 10 points for each part.

Algorithms and Problems

Algorithm — mechanical procedure that computes something (it can be executed by a computer)

Algorithms are used for solving **problems**.

An example of an algorithmic problem:

Input: Natural numbers x and y.

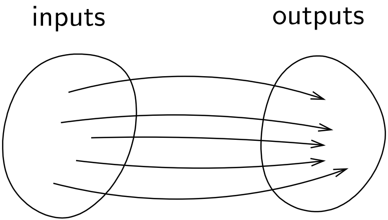
Output: Natural number z such that $z = x + y$.

Problems

Problem

When specifying a **problem** we must determine:

- what is the set of possible inputs
- what is the set of possible outputs
- what is the relationship between inputs and outputs



Examples of Problems

Problem "Sorting"

Input: A sequence of elements $a_1, a_2, \ldots, a_n$.

Output: Elements of the sequence $a_1, a_2, \ldots, a_n$ ordered from the least to the greatest.

Example:

- Input: $8, 13, 3, 10, 1, 4$
- Output: $1, 3, 4, 8, 10, 13$

Remark: A particular input of a problem is called an **instance** of the problem.

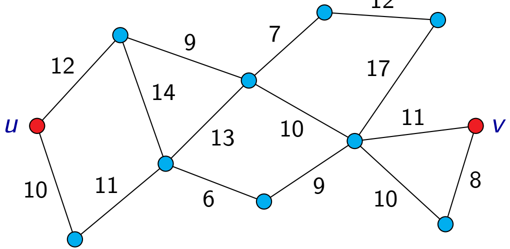
Examples of Problems

Problem "'Finding the shortest path in an (undirected) graph'

Input: An undirected graph $G = (V, E)$ with edges labelled with numbers, and a pair of nodes $u, v \in V$.

Output: The shortest path from node u to node v.

Example:



Algorithms and Problems

An algorithm **solves** a given problem if:

- For each input, the computation of the algorithm halts after a finite number of steps.
- For each input, the algorithm produces a correct output.

Correctness of an algorithm — verifying that the algorithm really solves the given problem

Computational complexity of an algorithm:

- **time complexity** — how the running time of the algorithm depends on the size of input data
- **space complexity** — how the amount of memory used by the algorithm depends on the size of input data

Algorithms and Problems

Theoretical computer science overlaps with many other areas of mathematics and computer science:

- graph theory
- number theory
- cryptography
- computational geometry
- searching in text, data compression
- game theory
- . . .

Other Examples of Problems

Problem "Primality"

Input: A natural number n.

Output: YES if n is a prime, NO otherwise.

Remark: A natural number n is a **prime** if it is greater than 1 and is divisible only by numbers 1 and n.

Few of the first primes: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$

Decision Problems

The problems, where the set of outputs is $\{\textsc{Yes}, \textsc{No}\}$ are called **decision problems**.

Decision problems are usually specified in such a way that instead of describing what the output is, a question is formulated.

Example:

Problem "Primality"

Input: A natural number n.

Question: Is n a prime?

Algorithmically Solvable Problems

Let us assume we have a problem P.

If there is an algorithm solving the problem P then we say that the problem P is **algorithmically solvable**.

If P is a decision problem and there is an algorithm solving the problem P then we say that the problem P is **decidable (by an algorithm)**.

If we want to show that a problem P is algorithmically solvable, it is sufficient to show some algorithm solving it (and possibly show that the algorithm really solves the problem P).

Algorithmically Unsolvable Problems

A problem that is not algorithmically solvable is **algorithmically unsolvable**.

A decision problem that is not decidable is **undecidable**.

Surprisingly, there are many (exactly defined) problems, for which it was proved that they are not algorithmically solvable.

Computability theory — area of theoretical computer science studying, which problems can be solved algorithmically and which cannot.

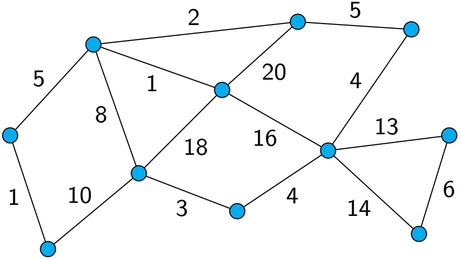
Complexity Theory

Many problems are algorithmically solvable but there do not exist (or are not known) efficient algorithms solving them:

TSP - traveling salesman problem

Input: An undirected graph G with edges labelled with natural numbers.

Output: A shortest closed path that goes through all vertices of the graph.



Theoretical Computer Science

Some other areas of theoretical computer science:

- complexity theory
- parallel and distributed algorithms
- models of computation

Theory of Formal Languages

An area of theoretical computer science dealing with questions concerning **syntax**.

- **Language** — a set of words
- **Word** — a sequences of symbols from some alphabet
- **Alphabet** — a set of **symbols** (or **letters**)

Formalisms used for description of languages:

- grammars
- regular expressions
- automata

Logic

Logic — study of reasoning and argumentation

- formalization of propositions in a natural language using **formulas**
- study where a conclusion follows from assumptions
- questions concerning proofs and provability
- connected with study of foundations of mathematics and set theory

Two most important types of logic:

- propositional logic
- predicate logic

Propositional Logic

Logical Inference

- *If the train arrives late and there are no taxis at the station, then John is late for his meeting.*
- *John is not late for his meeting.*
- *The train did arrive late.*

- *There were taxis at the station.*

- *If it is raining and Jane does not have her umbrella with her, then she will get wet.*
- *Jane is not wet.*
- *It is raining.*

- *Jane has her umbrella with her.*

Logical Inference

p	The train is late.	It is raining.
q	There are taxis at the station.	Jane has her umbrella with her.
r	John is late for his meeting.	Jane gets wet.

If p and not q, then r.

Not r.

p.

---

q.

Propositions

Examples of propositions:

- *"Jane gets wet."*
- *"If it is raining and Jane does not have her umbrella with her, then she will get wet."*
- *"Paris is the capital of Japan."*
- *"There are infinitely many primes."*
- "$1 + 1 = 3$"
- *"Number* $\sqrt{2}$ *is irrational."*

Examples of formulations, which are not propostions:

- *"The sum of numbers* 2 *and* 5*."*
- *"At what time a train goes from Ostrava to Prague today."*
- *"The distance of Earth from the Sun."*

Propositions

Examples of formulation, which are not propositions is a strict sense, but from which we can obtain propositions by assigning values to variables:

- *"$x > 5$"*
- *"$x + y = z$"*
- *"x is the greatest element of set A"*

But these are propositions:

- *"There exists a natural number x such that $x > 5$."*
- *"For each natural number x it holds that $x > 5$."*

Logical Connectives

Atomic proposition — it cannot be decomposed into simpler propositions

"It is raining."

Compound proposition — it is composed from some simpler propositions

"If it is raining and Jane does not have her umbrella with her, then she will get wet."

It consists of propositions:

- *"It is raining."*
- *"Jane has her umbrella with her."*
- *"Jane gets wet."*

Logical Connectives

Representation of propositions using **formulas**:

Symbol	Log. connective	Example of use	Informal meaning
$\neg$	negation	$\neg p$	*"not p"*
$\wedge$	conjunction	$p \wedge q$	*"p and q"*
$\vee$	disjunction	$p \vee q$	*"p or q"*
$\rightarrow$	implication	$p \rightarrow q$	*"if p then q"*
$\leftrightarrow$	equivalence	$p \leftrightarrow q$	*"p if and only if q"*

Atomic propositions — p, q, r, . . .
(possibly with indexes — p_0, p_1, p_2, . . .)

Logical Connectives

"If it is raining and Jane does not have her umbrella with her, then she will get wet."

The proposition written as a formula:

$$(p \wedge \neg q) \rightarrow r$$

Atomic propositions:

- p — *"It is raining."*
- q — *"Jane has her umbrella with her."*
- r — *"Jane gets wet."*

Truth Values

true	false
1	0
T	F
yes	no

We will use 0 and 1 to denote the truth values.

The truth values are also called **boolean** values.

Negation

The **negation** of a proposition φ is the proposition *"not φ"*, or the proposition *"it is not the case that φ"*. For example, the negation of

"the number 5 is a prime"

is a proposition

"it is not true that the number 5 is a prime"

or

"the number 5 is not a prime".

In formulas, negation is denoted by symbol "$\neg$".

Formally: $\neg\varphi$ example: $\neg p$

φ	$\neg\varphi$
0	1
1	0

Negation

Example: The negation of proposition

"Charles is the tallest boy in the class"

is proposition

"Charles is not the tallest boy in the class"

or

"it is not the case that Charles is the tallest boy in the class".

But this proposition is not negation of the previous proposition:

"Charles is the smallest boy in the class".

Negation

Example:

"It is not the case that the number 5 is not positive."

can be formalized as

$$\neg\neg p \qquad \text{or} \qquad \neg(\neg p)$$

Similarly

"it is not the case that it is not the case that the number 5 is not positive"

can be written as

$$\neg\neg\neg p \qquad \text{or} \qquad \neg(\neg(\neg p))$$

p — *"the number 5 is positive"*

Conjunction

The **conjunction** of propositions φ and ψ is proposition *"φ and ψ"*.

Example: The conjuction of propositions *"Copenhagen is the capital of Denmark"* and "$2 + 2 = 4$" is proposition

"Copenhagen is the capital of Denmark and $2 + 2 = 4$."

In formulas, conjuction is denoted by symbol "$\wedge$".

$$p \wedge q$$

- p — *"Copenhagen is the capital of Denmark"*
- q — "$2 + 2 = 4$"

Conjunction

φ	ψ	$\varphi \wedge \psi$
0	0	0
0	1	0
1	0	0
1	1	1

Examples of false propositions:

- *"Helsinki are the capital of Italy and Charles University was founded in 1348."*
- *"Asia is the largest continent and* $3 + 5 = 14$*."*
- *"There are only finitely many primes and Pilsen is the capital of USA."*

Conjunction

Example:

"I wanted to arrive but I have not catched the train"

can be formalized as $p \wedge \neg q$.

Example:

"the number 12 is divisible by 3 and 4"

can be formalized as $p \wedge q$, where

- p — *"the number 12 is divisible by 3"*
- q — *"the number 12 is divisible by 4"*

Conjunction

Examples of propositions where “and” is **not** used as conjunction:

- *“The left and righ bank of the river are connected by a bridge.”*
- *“The number 3 is the greatest common divisor of 12 and 15.”*
- *“Point A lies at the intersection of lines p_1 and p_2.”*

Conjunction

Proposition *"not φ, nor ψ"* claims the same thing as

"not φ and not ψ".

This can be formalized as

$$\neg\varphi \wedge \neg\psi$$

Example: Proposition *"line p does not go through point A, nor point B"* can be formalized as

$$\neg r \wedge \neg s\,,$$

where

- r — *"line p goes through point A"*
- s — *"line p goes through point B"*

Disjunction

The **disjunction** of propositions φ and ψ is proposition "φ *or* ψ".

Example: The disjunction of propositions *"whales are mammals"* and *"Czeck Republic is in Europe"* is proposition

"whales are mammals or Czech Republic is in Europe".

In formulas, disjunction is denoted by symbol "$\vee$".

$$p \vee q$$

- p — *"whales are mammals"*
- q — *"Czech Republic is in Europe"*

Disjunction

Disjunction is *"or"* in **non-exclusive** sense.

φ	ψ	$\varphi \vee \psi$
0	0	0
0	1	1
1	0	1
1	1	1

Implication

Implication — *"if* φ *then* ψ*"*

- φ — assumption (hypothesis)
- ψ — conclusion

Example:

> *" If Peter was well-prepared for the exam, he obtained a good grade."*

Implication is denoted by symbol "$\rightarrow$".

$$p \rightarrow q$$

- p — *"Peter was well-prepared for the exam"*
- q — *"Peter obtained a good grade"*

Implication

φ	ψ	$\varphi \rightarrow \psi$
0	0	1
0	1	1
1	0	0
1	1	1

Remark: Formula $p \rightarrow q$ is true exactly in those cases where the following formula is true:

$$\neg p \vee q$$

Implication

Implication does **not** express causal dependence.

Example:

- *"If Washington is the capital of USA, then* $1 + 1 = 2$*."*
- *"If Washington is the capital of USA, then* $1 + 1 = 3$*."*
- *"If Tokyo is the capital of USA, then* $1 + 1 = 2$*."*
- *"If Tokyo is the capital of USA, then* $1 + 1 = 3$*."*

Implication

Implication $p \longrightarrow q$ can be expressed in a natural language in many different ways:

- *“q if p”*
- *“p only if q”*
- *“p implies q”*
- *“q provided that p”*
- *“from p follows q”*
- *“p is a sufficient condition for q”*
- *“q is a necessary condition for p”*

Implication

If $\varphi \rightarrow \psi$ is true and also φ is true, we can infer from this that also ψ is true.

Example: When it holds that

- *"if today is Tuesday, then tomorrow is Wednesday"*
- *"today is Tuesday"*

we can infer from this that

- *"tomorrow is Wednesday"*

Equivalence

Equivalence — "φ *if and only if* ψ"

Example:

> *"Triangle ABC has all three sides of the same length if and only if it has all three angles of the same size."*

The logical connective equivalence is denoted by symbol "$\leftrightarrow$"

$$p \leftrightarrow q$$

- p — *"triangle ABC has all three sides of the same length"*
- q — *"triangle ABC has all three angles of the same size"*

Equivalence

φ	ψ	$\varphi \leftrightarrow \psi$
0	0	1
0	1	0
1	0	0
1	1	1

Remark: Formula $p \leftrightarrow q$ says basically the same thing as

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

Equivalence

Alternatives for expressing equivalence $p \leftrightarrow p$:

- *"p is a necessary and sufficient condition for q"*
- *"p iff q"*

Equivalence is often used in **definitions** of new notions:

Example:

- *"A triangle is isoscele if and only if at least two of its sides are equal in length."*
- *"A triangle is isoscele if at least two of its sides are equal in length."*

Formulas of Propositional Logic

- **Syntax** — what are well-formed formulas of propositional logic
- **Semantics** — assigns meaning to formulas and to individual symbols occurring in these formulas

Syntax of Formulas of Propositional Logic

Formulas — sequences of symbols from a given **alphabet**:

- **atomic propositions** — for example symbols "p", "q", "r", etc.
- **logical connectives** — symbols "$\neg$", "$\wedge$", "$\vee$" "$\rightarrow$", "$\leftrightarrow$"
- **parentheses** — symbols "$($" and "$)$"

Not every sequence of these symbols is a formula.

For example, this is not a formula:

$$\wedge\vee)p\neg((\neg$$

Syntax of Formulas of Propositional Logic

Definition

Well-formed **formulas of propositional logic** are sequences of symbols constructed according to the following rules:

1. If p is an atomic proposition, then p is a well-formed formula.
2. If φ and ψ are well-formed formulas, then also $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ a $(\varphi \leftrightarrow \psi)$ are well-formed formulas.
3. There are no other well-formed formulas than those constructed according to two previous rules.

Syntax of Formulas of Propositional Logic

Examples of well-formed formulas:

- q
- $(\neg q)$
- r
- $((\neg q) \rightarrow r)$
- p
- $(p \leftrightarrow r)$
- $(\neg(p \leftrightarrow r))$
- $(((\neg q) \rightarrow r) \wedge (\neg(p \leftrightarrow r)))$

An example of a sequnce of symbols, which is not a well-formed formula:

- $(p \wedge \vee q)$

Syntax of Formulas of Propositional Logic

Formula ψ is a **subformula** of formula φ if at least one of the following possibilities holds:

- Formula ψ is the same formula as formula φ.
- If formula φ is of the form $(\neg\chi)$, then ψ is a subformula of formula χ.
- If formula φ is of the form $(\chi_1 \wedge \chi_2)$, $(\chi_1 \vee \chi_2)$, $(\chi_1 \rightarrow \chi_2)$, or $(\chi_1 \leftrightarrow \chi_2)$, then ψ is a subformula of formula χ_1 or a subformula of formula χ_2.

Example: Subformulas of formula $((\neg(p \wedge q)) \leftrightarrow r)$:

$$p \qquad q \qquad r \qquad (p \wedge q) \qquad (\neg(p \wedge q)) \qquad ((\neg(p \wedge q)) \leftrightarrow r)$$

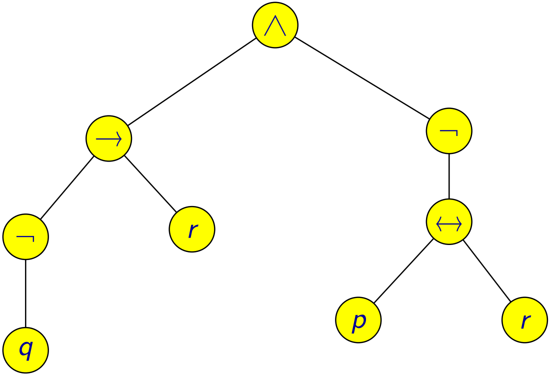
Syntax of Formulas of Propositional Logic

Alternative symbols for logical connectives:

Connective	Symbol	Alternative symbols
negation	$\neg$	$\sim$
conjunction	$\wedge$	$\&$
implication	$\rightarrow$	$\Rightarrow$, $\supset$
equivalence	$\leftrightarrow$	$\Leftrightarrow$, $\equiv$

Syntax of Formulas of Propositional Logic

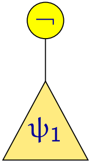
An abstract syntax tree of formula $(((\neg q) \rightarrow r) \wedge (\neg(p \leftrightarrow r)))$:



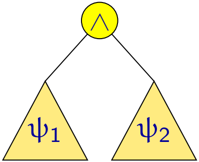
Syntax of Formulas of Propositional Logic

p

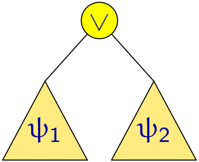
$(\neg\psi_1)$



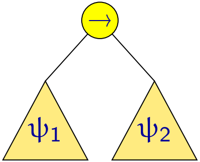
$(\psi_1 \wedge \psi_2)$



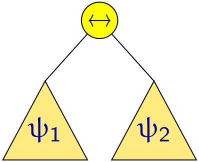
$(\psi_1 \vee \psi_2)$



$(\psi_1 \rightarrow \psi_2)$



$(\psi_1 \leftrightarrow \psi_2)$



Syntax of Formulas of Propositional Logic

Arity of logical connectives:

- **unary connective** (arity 1): $\neg$
- **binary connectives** (arity 2): $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

Syntax of Formulas of Propositional Logic

Conventions for omitting parentheses:

- Outermost pair of parentheses can be omitted.
- Priority of logical connectives (from the highest to the lowest):

$$\neg \qquad \wedge \qquad \vee \qquad \rightarrow \qquad \leftrightarrow$$

- Instead of $\neg(\neg\varphi)$, it is possible to write $\neg\neg\varphi$.

Example: Instead of $((\neg p) \wedge (r \rightarrow (q \vee s)))$, it is possible to write

$$\neg p \wedge (r \rightarrow q \vee s)$$

Remark: Other conventions will be described later.

Semantics of Propositional Logic

At — a set of atomic propositions

For example

- $\mathit{At} = \{p, q, r\}$, or
- $\mathit{At} = \{p_0, p_1, p_2, \ldots\}$

Definition

A **truth valuation** is an assignment of truth values (i.e., values from the set $\{0, 1\}$) to all atomic propositions from the set At.
(Formally, a truth valuation can be defined as a function $v : \mathit{At} \to \{0, 1\}$.)

Example: A truth valuation v for $\mathit{At} = \{p, q, r\}$, where

$$v(p) = 1 \qquad v(q) = 0 \qquad v(r) = 1$$

Semantics of Propositional Logic

If set At is finite and contains n atomic propositions, then there are 2^n truth valuations.

Example: $At = \{p, q, r\}$

$$\begin{array}{llll}
v_0: & v_0(p) = 0, & v_0(q) = 0, & v_0(r) = 0 \\
v_1: & v_1(p) = 0, & v_1(q) = 0, & v_1(r) = 1 \\
v_2: & v_2(p) = 0, & v_2(q) = 1, & v_2(r) = 0 \\
v_3: & v_3(p) = 0, & v_3(q) = 1, & v_3(r) = 1 \\
v_4: & v_4(p) = 1, & v_4(q) = 0, & v_4(r) = 0 \\
v_5: & v_5(p) = 1, & v_5(q) = 0, & v_5(r) = 1 \\
v_6: & v_6(p) = 1, & v_6(q) = 1, & v_6(r) = 0 \\
v_7: & v_7(p) = 1, & v_7(q) = 1, & v_7(r) = 1
\end{array}$$

Semantics of Propositional Logic

At truth valuation v, formula φ has truth value 1:

$$v \models \varphi$$

At truth valuation v, formula φ has truth value 0:

$$v \not\models \varphi$$

Semantics of Propositional Logic

Definition

Truth values of formulas of propositional logic in a given truth valuation v are defined as follows:

- For atomic proposition p, $v \models p$ iff $v(p) = 1$. (So, if $v(p) = 0$, then $v \not\models p$.)
- $v \models \neg\varphi$ iff $v \not\models \varphi$.
- $v \models \varphi \wedge \psi$ iff $v \models \varphi$ and $v \models \psi$.
- $v \models \varphi \vee \psi$ iff $v \models \varphi$ or $v \models \psi$.
- $v \models \varphi \rightarrow \psi$ iff $v \not\models \varphi$ or $v \models \psi$.
- $v \models \varphi \leftrightarrow \psi$ iff $v \models \varphi$ and $v \models \psi$, or $v \not\models \varphi$ and $v \not\models \psi$.

Semantics of Propositional Logic

φ	$\neg\varphi$
0	1
1	0

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Semantics of Propositional Logic

Example: $\mathcal{At} = \{p, q, r\}$
valuation v, where $v(p) = 1$, $v(q) = 0$, and $v(r) = 1$

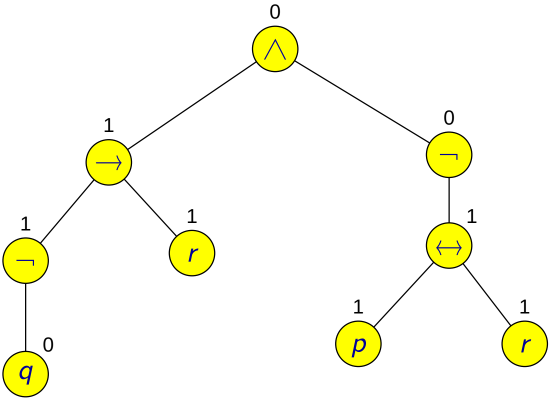
- $v \not\models q$
- $v \models \neg q$
- $v \models r$
- $v \models \neg q \rightarrow r$
- $v \models p$
- $v \models p \leftrightarrow r$
- $v \not\models \neg(p \leftrightarrow r)$
- $v \not\models (\neg q \rightarrow r) \wedge \neg(p \leftrightarrow r)$

p	q	r	$\neg q$	$\neg q \rightarrow r$	$p \leftrightarrow r$	$\neg(p \leftrightarrow r)$	φ
1	0	1	1	1	1	0	0

$\varphi := (\neg q \rightarrow r) \wedge \neg(p \leftrightarrow r)$

Semantics of Propositional Logic

$(\neg q \rightarrow r) \wedge \neg(p \leftrightarrow r)$



Semantics of Propositional Logic

$\varphi := (\neg q \rightarrow r) \wedge \neg(p \leftrightarrow r)$

p	q	r	$\neg q$	$\neg q \rightarrow r$	$p \leftrightarrow r$	$\neg(p \leftrightarrow r)$	φ
0	0	0	1	0	1	0	0
0	0	1	1	1	0	1	1
0	1	0	0	1	1	0	0
0	1	1	0	1	0	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	0	0
1	1	0	0	1	0	1	1
1	1	1	0	1	1	0	0

Those valuations, where the given formula is true, are called its **models**:

$$\begin{array}{llll} v_1: & v_1(p) = 0, & v_1(q) = 0, & v_1(r) = 1, \\ v_3: & v_3(p) = 0, & v_3(q) = 1, & v_3(r) = 1, \\ v_6: & v_6(p) = 1, & v_6(q) = 1, & v_6(r) = 0, \end{array}$$

Tautologies

Definition

Formula φ is a **tautology** if $v \models \varphi$ holds for every truth valuation v (i.e., if φ is true in every valuation).

Example: *“If it is raining, then it is raining.”*

$$p \rightarrow p$$

Example: *“It is Friday today, or it is not Friday today.”*

$$q \vee \neg q$$

Tautologies

An example of a more complicated tautology:

$$(p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$$

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$\neg p$	$(p \rightarrow \neg q) \rightarrow \neg p$	φ
0	0	1	1	1	1	1	1
0	1	1	0	1	1	1	1
1	0	0	1	1	0	0	1
1	1	1	0	0	0	1	1

Tautologies

Quite important are tautologies of the form $\varphi \rightarrow \psi$ or $\varphi \leftrightarrow \psi$
— they can be used for logical inference:

- If $\varphi \rightarrow \psi$ holds and φ holds, then also ψ must hold.

  In particular, if $\varphi \rightarrow \psi$ is a tautology and φ holds, we can deduce that also ψ holds.

  Example: $(p \wedge q) \rightarrow p$ is a tautology.

  If $p \wedge q$ holds, then also p holds.

  Example: $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$ is a tautology.

  If $p \rightarrow q$ holds and $\neg q$ holds, then $\neg p$ holds.

Tautologies

- If $\varphi \leftrightarrow \psi$ holds and φ holds, then ψ must hold.

  Similarly, if $\varphi \leftrightarrow \psi$ holds and ψ holds, then φ must hold.

  Example: $(\neg p \rightarrow q) \leftrightarrow (q \vee p)$ is a tautology.

  - If $\neg p \rightarrow q$ holds, then also $q \vee p$ must hold.
  - If $q \vee p$ holds, then also $\neg p \rightarrow q$ must hold.

Tautologies

When we take a tautology φ and replace all atomic propositions by arbitrary formulas, we obtain a tautology by this replacement.

Example: Formula $p \rightarrow (p \vee q)$ is a tautology.

This means that

$$\psi \rightarrow (\psi \vee \chi)$$

is a tautology for arbitrary formulas ψ and χ.

Replacement of atomic propositions:

- p is replaced with $q \vee \neg(r \rightarrow \neg s)$
- q is replaced with $\neg\neg(q \leftrightarrow p)$

We obtain tautology

$$(q \vee \neg(r \rightarrow \neg s)) \rightarrow ((q \vee \neg(r \rightarrow \neg s)) \vee \neg\neg(q \leftrightarrow p))$$

Contradictions

Definition

A formula φ is a **contradiction** if $v \not\models \varphi$ holds for every truth valuation v (i.e., when φ is false in every valuation).

Example: *"It is Wednesday today, and it is not Wednesday today."*

$$p \wedge \neg p$$

- φ is a tautology iff $\neg\varphi$ is a contradiction
- φ is a contradiction iff $\neg\varphi$ is a tautology

Satisfiable Formulas

Definition

A formula φ is **satisfiable** if there is at least one truth valuation v such that $v \models \varphi$.

- A formula is satisfiable iff it is not a contradiction.
- Every tautology is satisfiable but not every satisfiable formula is a tautology.

Example: A formula, which is satisfiable but not a tautology:

$$(p \vee q) \rightarrow p$$

- For example in valuation v_1, where $v_1(p) = 1$ and $v_1(q) = 0$, the formula is true.
- In valuation v_2, where $v_2(p) = 0$ and $v_2(q) = 1$, it is false.

Satisfiable Formulas

- φ is a tautology iff $\neg\varphi$ is not satisfiable
- φ is satisfiable iff $\neg\varphi$ is not a tautology

- **Satisfiable formulas**:
  - To show that a formula **is** satisfiable, it is sufficient to find a valuation, in which the formula is true.
  - To show that a formula **is not** satisfiable, it necessary to show that there is no valuation, in which the formula is true.

Tautologies and Contradictions

- **Tautologies**:
  - To show that a formula **is not** a tautology, it is sufficient to find a valuation, in which the formula is false.
  - To show that a formula **is** a tautology, it necessary to show that there is no valuation, in which the formula is false.
- **Contradictions**:
  - To show that formula **is not** a contradiction, it is sufficient to find a valuation, in which the formula is true.
  - To show that a formula **is** a contradiction, it necessary to show that there is no valuation, in which the formula is true.

Semantic Contradiction

For decing whether a formula φ is or is not a tautology (resp. a contradiction, satisfiable), the table method can be used.

It is usually not necessary to construct whole table. It is sufficient to concentrate on "interesting" cases.

- We can draw the syntax tree for the given formula and try to assign values 0 and 1 to its nodes.

For example, for deciding if a formula is a tautology:

- We start by assigning value 0 to the root.
- Then we assign values to other nodes, which are enforced by already assigned values.
- If we succeed in labelling whole tree consistently, we have a valuation, in which the formula is false.
- If we find out that there is no such valuation, then the formula is a tautology.

Semantic Contradiction

- Identical subtrees must be labelled identically — the same subformulas must have the same truth values.
- If some node must be labelled by both 0 and 1, we have a **contradiction** — no such valuation can exist.
- Sometimes it is necessary to return back and try several possible assigned values — in cases, where the already assigned values do not enforce a truth value for a node, to which no value has been assigned yet.

Semantic Contradiction

Example:

- $(p \rightarrow (q \vee r)) \vee (p \rightarrow r)$ (not a tautology)

  $\langle$*a solution on a whiteboard*$\rangle$

- $(p \leftrightarrow (\neg q \vee r)) \rightarrow (\neg p \rightarrow q)$ (a tautology)

  $\langle$*a solution on a whiteboard*$\rangle$

- $((p \wedge q) \vee (\neg p \wedge \neg q)) \vee (\neg p \wedge q)$ (not a tautology)

  $\langle$*a solution on a whiteboard*$\rangle$

Equivalence of Formulas

Definition

Formulas φ and ψ are **logically equivalent** if for each truth valuation v it holds that φ and ψ have the same truth value in valuation v, i.e.,

$$v \models \varphi \qquad \text{iff} \qquad v \models \psi.$$

The fact that formulas φ and ψ are logically equivalent is denoted

$$\varphi \; \Leftrightarrow \; \psi.$$

Formulas φ and ψ are logically equivalent iff $\varphi \leftrightarrow \psi$ is a tautology.

Equivalence of Formulas

Example: $\neg(p \rightarrow q) \;\Leftrightarrow\; p \wedge \neg q$

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg q$	$p \wedge \neg q$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

Equivalence of Formulas

To show that formulas φ and ψ are **not** equivalent, it is sufficient to find a valuation v such that:

- $v \models \varphi$ and $v \not\models \psi$, or
- $v \not\models \varphi$ and $v \models \psi$.

Example: $p \vee (q \wedge r)$ is not equivalent to $(p \vee q) \wedge r$

Valuation v, where:

- $v(p) = 1$
- $v(q) = 1$
- $v(r) = 0$

In this valuation, $p \vee (q \wedge r)$ holds but $(p \vee q) \wedge r$ does not hold.

Some Important Equivalences

- Equivalences for negation:

$\neg\neg p \;\Leftrightarrow\; p$	*double negation*

- Equivalences for conjunction:

$(p \wedge q) \wedge r \;\Leftrightarrow\; p \wedge (q \wedge r)$	*associativity*
$p \wedge q \;\Leftrightarrow\; q \wedge p$	*commutativity*
$p \wedge p \;\Leftrightarrow\; p$	*idempotence*

- Equivalences for disjunction:

$(p \vee q) \vee r \;\Leftrightarrow\; p \vee (q \vee r)$	*associativity*
$p \vee q \;\Leftrightarrow\; q \vee p$	*commutativity*
$p \vee p \;\Leftrightarrow\; p$	*idempotence*

Some Important Equivalences

- Distributivity of $\wedge$ and $\vee$:

$$p \wedge (q \vee r) \;\Leftrightarrow\; (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \;\Leftrightarrow\; (p \vee q) \wedge (p \vee r)$$

- De Morgan's laws:

$$\neg(p \wedge q) \;\Leftrightarrow\; \neg p \vee \neg q$$
$$\neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q$$

- Equivalences for implication:

$$p \rightarrow q \;\Leftrightarrow\; \neg p \vee q$$
$$\neg(p \rightarrow q) \;\Leftrightarrow\; p \wedge \neg q$$

Some Important Equivalences

– Equivalences for $\leftrightarrow$:

$$
\begin{array}{lll}
(p \leftrightarrow q) \leftrightarrow r \;\Leftrightarrow\; p \leftrightarrow (q \leftrightarrow r) & \qquad & \textit{associativity} \\
p \leftrightarrow q \;\Leftrightarrow\; q \leftrightarrow p & & \textit{commutativity} \\
p \leftrightarrow q \;\Leftrightarrow\; (p \rightarrow q) \wedge (q \rightarrow p) & & \\
p \leftrightarrow q \;\Leftrightarrow\; (p \vee \neg q) \wedge (\neg p \vee q) & & \\
p \leftrightarrow q \;\Leftrightarrow\; (p \wedge q) \vee (\neg p \wedge \neg q) & &
\end{array}
$$

Equivalence of Formulas

Let us assume that formulas φ and ψ are logically equivalent, i.e.,

$$\varphi \Leftrightarrow \psi .$$

If we replace atomic propositions in φ and ψ by arbitrary formulas, we obtain again a pair of equivalent formulas.

Example: $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Therefore, for arbitrary formulas χ_1 and χ_2 is

$$\neg(\chi_1 \vee \chi_2) \Leftrightarrow \neg\chi_1 \wedge \neg\chi_2$$

Equivalence of Formulas

$$\neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q$$

Replacement of atomic propositions:

- p replaced by $q \vee \neg(r \rightarrow \neg s)$
- q replaced by $\neg(q \leftrightarrow p)$

We obtain

$$\neg((q \vee \neg(r \rightarrow \neg s)) \vee \neg(q \leftrightarrow p)) \;\Leftrightarrow\; \neg(q \vee \neg(r \rightarrow \neg s)) \wedge \neg\neg(q \leftrightarrow p)$$

Equivalence of Formulas

Let us assume that φ is a formula and ψ its subformula.

If we replace some occurrence of subformula ψ in formula φ with a formula ψ' such that $\psi \Leftrightarrow \psi'$, we obtain a formula φ' such that

$$\varphi \Leftrightarrow \varphi' .$$

Example: In formula

$$\neg((p \rightarrow q) \vee (\neg(p \rightarrow q) \rightarrow r))$$

we replace the second occurrence of subformula $p \rightarrow q$ with an equivalent formula $\neg p \vee q$.

We obtain

$$\neg((p \rightarrow q) \vee (\neg(\neg p \vee q) \rightarrow r))$$

Equivalent Transformations

For arbitrary formulas φ, ψ, and χ, it holds:

- $\varphi \Leftrightarrow \varphi$.
- If $\varphi \Leftrightarrow \psi$, then $\psi \Leftrightarrow \varphi$.
- If $\varphi \Leftrightarrow \psi$ and $\psi \Leftrightarrow \chi$, then $\varphi \Leftrightarrow \chi$.

When we try to prove equivalence of formulas, we can proceed by smaller steps:

For example, if it holds that $\varphi_1 \Leftrightarrow \varphi_2$, $\varphi_2 \Leftrightarrow \varphi_3$, $\varphi_3 \Leftrightarrow \varphi_4$, and $\varphi_4 \Leftrightarrow \varphi_5$, we can conclude that

$$\varphi_1 \Leftrightarrow \varphi_5 .$$

This can be written as

$$\varphi_1 \Leftrightarrow \varphi_2 \Leftrightarrow \varphi_3 \Leftrightarrow \varphi_4 \Leftrightarrow \varphi_5$$

Equivalent Transformations

Example: The proof that

$$(p \wedge q) \rightarrow r \;\Leftrightarrow\; p \rightarrow (q \rightarrow r)$$

$$\begin{aligned}
(p \wedge q) \rightarrow r \;&\Leftrightarrow\; \neg(p \wedge q) \vee r\\
&\Leftrightarrow\; (\neg p \vee \neg q) \vee r\\
&\Leftrightarrow\; \neg p \vee (\neg q \vee r)\\
&\Leftrightarrow\; \neg p \vee (q \rightarrow r)\\
&\Leftrightarrow\; p \rightarrow (q \rightarrow r)
\end{aligned}$$

Equivalent Transformations

Every formula can be transformed to an equivalent formula that uses only "$\neg$", "$\wedge$", and "$\vee$" as logical connectives.

- The connective "$\leftrightarrow$" can be replaced by other connectives using the following equivalences:
  - $p \leftrightarrow q \;\Leftrightarrow\; (p \rightarrow q) \wedge (q \rightarrow p)$
  - $p \leftrightarrow q \;\Leftrightarrow\; (p \vee \neg q) \wedge (\neg p \vee q)$
  - $p \leftrightarrow q \;\Leftrightarrow\; (p \wedge q) \vee (\neg p \wedge \neg q)$
- The connective "$\rightarrow$" can be replaced by "$\neg$" and "$\vee$" using the following equivalence:
  - $p \rightarrow q \;\Leftrightarrow\; \neg p \vee q$

Example:

$$\begin{aligned}(\neg q \rightarrow r) \wedge \neg(p \leftrightarrow r) &\Leftrightarrow (\neg\neg q \vee r) \wedge \neg(p \leftrightarrow r)\\ &\Leftrightarrow (\neg\neg q \vee r) \wedge \neg((p \wedge r) \vee (\neg p \wedge \neg r))\end{aligned}$$

Equivalent Transformations

Every formula can be transformed to an equivalent formula, which contains only logical connectives "$\neg$", "$\wedge$" and "$\vee$", and where negations are applied only to atomic propositions.

- We can assume that formula contains only "$\neg$", "$\wedge$" and "$\vee$".
- Negations can be "pushed" to atomic propositions using the following equivalences:
  - $\neg\neg p \;\Leftrightarrow\; p$
  - $\neg(p \wedge q) \;\Leftrightarrow\; \neg p \vee \neg q$
  - $\neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q$

Equivalent Transformations

Example:

$$
\begin{aligned}
&(\neg\neg q \lor r) \land \neg((p \land r) \lor (\neg p \land \neg r)) \\
&\quad\Leftrightarrow\ (q \lor r) \land \neg((p \land r) \lor (\neg p \land \neg r)) \\
&\quad\Leftrightarrow\ (q \lor r) \land (\neg(p \land r) \land \neg(\neg p \land \neg r)) \\
&\quad\Leftrightarrow\ (q \lor r) \land ((\neg p \lor \neg r) \land \neg(\neg p \land \neg r)) \\
&\quad\Leftrightarrow\ (q \lor r) \land ((\neg p \lor \neg r) \land (\neg\neg p \lor \neg\neg r)) \\
&\quad\Leftrightarrow\ (q \lor r) \land ((\neg p \lor \neg r) \land (p \lor \neg\neg r)) \\
&\quad\Leftrightarrow\ (q \lor r) \land ((\neg p \lor \neg r) \land (p \lor r))
\end{aligned}
$$

Logical Constants

For some purposes it can be useful to introduce the following special formulas:

- $\top$ — a formula, which is always true
- $\bot$ — a formula, which is always false

For every truth valuation v it holds:

- $v \models \top$ ($\top$ has always truth value 1)
- $v \not\models \bot$ ($\bot$ has always truth value 0)

Logical Constants

Symbols $\top$ and $\bot$ can be viewed as abbreviations:

- $\top$ stands for an arbitrary tautology (e.g., $p \rightarrow p$)
- $\bot$ stands for an arbitrary contradiction (e.g., $p \wedge \neg p$)

Alternatively, we could extend the definition of syntax and semantics of propositional logic.

Symbols $\top$ and $\bot$ can be viewed as logical connectives with arity 0.

Logical Constants

Examples of equivalences that hold for $\top$ and $\bot$ (and for arbitrary p):

$$\begin{aligned} \top &\Leftrightarrow p \vee \neg p \\ \neg\top &\Leftrightarrow \bot \\ p \wedge \top &\Leftrightarrow p \\ p \vee \top &\Leftrightarrow \top \end{aligned} \qquad\qquad \begin{aligned} \bot &\Leftrightarrow p \wedge \neg p \\ \neg\bot &\Leftrightarrow \top \\ p \vee \bot &\Leftrightarrow p \\ p \wedge \bot &\Leftrightarrow \bot \end{aligned}$$

Equivalence of Formulas

It is **not** necessary for equivalent formulas to contain the same atomic propositions.

Example: $(q \rightarrow \neg\neg q) \wedge \neg p \;\Leftrightarrow\; p \rightarrow (r \wedge \neg r)$

$$\begin{aligned}
(q \rightarrow \neg\neg q) \wedge \neg p &\Leftrightarrow (q \rightarrow q) \wedge \neg p \\
&\Leftrightarrow \top \wedge \neg p \\
&\Leftrightarrow \neg p \\
&\Leftrightarrow \neg p \vee \bot \\
&\Leftrightarrow p \rightarrow \bot \\
&\Leftrightarrow p \rightarrow (r \wedge \neg r)
\end{aligned}$$

For example, also all tautologies are logically equivalent.

Conjunctions and Disjunctions of Several Formulas

Due to associativity of conjunction, it holds for example:

$$p \wedge ((q \wedge r) \wedge (s \wedge t)) \;\Leftrightarrow\; (p \wedge q) \wedge ((r \wedge s) \wedge t)$$

Both these formulas are also equivalent to formulas

- $p \wedge (q \wedge (r \wedge (s \wedge t)))$
- $(((p \wedge q) \wedge r) \wedge s) \wedge t$

All these formulas are true iff all propositions p, q, r, s, and t are true.

Convention: Due to associativity of conjunction, the parentheses can be omitted and we can write

$$p \wedge q \wedge r \wedge s \wedge t$$

Conjunctions and Disjunctions of Several Formulas

Because conjunction is not only associative but also commutative, the order of members of such more complicated conjunction is not important. For example:

$$r \wedge t \wedge q \wedge s \wedge p \quad \Leftrightarrow \quad p \wedge q \wedge r \wedge s \wedge t$$

Due to idempotence, also the number of occurrences of each member is not important.
For example:

$$p \wedge q \wedge p \quad \Leftrightarrow \quad q \wedge p \wedge q \wedge q$$

Conjunctions and Disjunctions of Several Formulas

The same holds also for disjunction, e.g.:

$$(p \vee q) \vee (r \vee q) \quad \Leftrightarrow \quad q \vee (p \vee (r \vee (r \vee r)))$$

Convention: Instead of $(p \vee q) \vee (r \vee (s \vee t))$ we can write

$$p \vee q \vee r \vee s \vee t$$

All this holds not only for atomic propositions but also for arbitrary formulas, e.g.:

- Instead of $(\varphi_1 \wedge \varphi_2) \wedge (\varphi_3 \wedge (\varphi_4 \wedge \varphi_5))$ we can write

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_5$$

Konjunkce a disjunkce více formulí

Conjunction of n formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$, where $n \geq 0$, is the formula

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$$

In particular:

- For $n = 0$, the conjunction is the formula $\top$.
- For $n = 1$, the conjunction is the formula φ_1.

Disjunction of n formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$, where $n \geq 0$, is the formula

$$\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$$

In particular:

- For $n = 0$, the disjunction is the formula $\bot$.
- For $n = 1$, the disjunction is the formula φ_1.

Conjunctions and Disjunctions of Several Formulas

Conjunction $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$:

- The whole formula is true iff all formulas φ_1, φ_2, . . . , φ_n are true.
- If some formula φ_i is equivalent to $\bot$, then the whole formula is equivalent to $\bot$.
- If some formula φ_i is equivalent to a negation of some formula φ_j (i.e., $\varphi_i \Leftrightarrow \neg\varphi_j$), then the whole formula is equivalent to $\bot$.
- If some formula φ_i is equivalent to $\top$, then it is possible to omit the formula φ_i from the whole formula.

Konjunkce a disjunkce více formulí

Disjunction $\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$:

- The whole formula is true iff at least one of formulas φ_1, φ_2, …, φ_n is true.
- If some formula φ_i is equivalent to $\top$, then the whole formula is equivalent to $\top$.
- If some formula φ_i is equivalent to a negation of some formula φ_j (i.e., $\varphi_i \Leftrightarrow \neg\varphi_j$), then the whole formula is equivalent to $\top$.
- If some formula φ_i is equilvalent to $\bot$, then it is possible to omit the formula φ_i from the whole formula.

Normal Forms of Fomulas

- **Literal** — an atomic proposition or its negation, e.g.,

$$p \qquad\qquad \neg q \qquad\qquad \neg r$$

- An **elementary conjunction** — a conjunction of one or more literals, e.g.,

$$(p \wedge \neg q) \qquad\qquad (r) \qquad\qquad (q \wedge \neg r \wedge p)$$

- An **elementary disjunction** (**clause**) — a disjunction of one or more literals, e.g.,

$$(p \vee \neg q) \qquad\qquad (r) \qquad\qquad (q \vee \neg r \vee p)$$

Normal Forms of Fomulas

Example:

- Elementary conjunction

  $$(p \wedge \neg q \wedge r \wedge \neg s \wedge \neg t)$$

  is **true** in exactly those truth valuations v where

  $v(p) = 1 \qquad v(q) = 0 \qquad v(r) = 1 \qquad v(s) = 0 \qquad v(t) = 0$

- Elementary disjunction

  $$(p \vee \neg q \vee r \vee \neg s \vee \neg t)$$

  is **false** in exactly those truth valuations v where

  $v(p) = 0 \qquad v(q) = 1 \qquad v(r) = 0 \qquad v(s) = 1 \qquad v(t) = 1$

Normal Forms of Fomulas

- **Disjunctive normal form** (**DNF**) — a disjunction of one or more elementary conjunctions, e.g.,

$$(p \wedge \neg q) \ \vee \ (\neg r) \ \vee \ (\neg r \wedge \neg p \wedge \neg q)$$

- **Conjunctive normal form** (**CNF**) — a conjunction of one or more elementary disjunctions (clauses), e.g.,

$$(p \vee \neg q) \ \wedge \ (\neg r) \ \wedge \ (\neg r \vee \neg p \vee \neg q)$$

Remark: Formulas $\bot$ and $\top$ will be also considered to be in DNF and CNF.

Normal Forms of Fomulas

Let us assume that formula φ in CNF contains no elemetary disjunction with literals p and $\neg p$ (such elementary disjunction can be omitted).

This formula φ is a **tautology** iff it is $\top$, i.e., if it contains no elementary disjunctions.

Let us assume that formula ψ in DNF contains no elementary conjunctions with literals p and $\neg p$ (such elementary conjunction can be omitted).

This formula ψ is a **contradiction** iff it is $\bot$, i.e., if it contains no elementary conjunctions.

Normal Forms of Fomulas

Transformation of a formula to DNF and CNF:

- We can assume that the formula contains only atomic propositions, connectives "$\neg$" applied to atomic propositions, and connectives "$\wedge$" and "$\vee$".
- The required form of the formula can be obtained by use of the following equivalences:
  - $p \wedge (q \vee r) \;\Leftrightarrow\; (p \wedge q) \vee (p \wedge r)$ — for transformation to DNF
  - $p \vee (q \wedge r) \;\Leftrightarrow\; (p \vee q) \wedge (p \vee r)$ — for transformation to CNF

Normal Forms of Fomulas

Example: Transformation of formula $q \wedge ((\neg p \vee \neg r) \wedge (p \vee r))$ to DNF:

$$
\begin{aligned}
& q \wedge ((\neg p \vee \neg r) \wedge (p \vee r)) \\
\Leftrightarrow\ & (q \wedge (\neg p \vee \neg r)) \wedge (p \vee r) \\
\Leftrightarrow\ & ((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge (p \vee r) \\
\Leftrightarrow\ & (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge p) \ \vee \ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (((q \wedge \neg p) \wedge p) \vee ((q \wedge \neg r) \wedge p)) \ \vee \ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (q \wedge \neg p \wedge p) \ \vee \ (q \wedge \neg r \wedge p) \ \vee \ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (q \wedge \bot) \ \vee \ (q \wedge \neg r \wedge p) \ \vee \ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & \bot \ \vee \ (q \wedge \neg r \wedge p) \ \vee \ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (q \wedge \neg r \wedge p) \ \vee \ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee \ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & \quad \ldots
\end{aligned}
$$

Normal Forms of Fomulas

$$
\begin{aligned}
&\dots\\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee\ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r)\\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee\ (((q \wedge \neg p) \wedge r) \ \vee\ ((q \wedge \neg r) \wedge r))\\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee\ (q \wedge \neg p \wedge r) \ \vee\ (q \wedge \neg r \wedge r)\\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee\ (\neg p \wedge q \wedge r) \ \vee\ (q \wedge \neg r \wedge r)\\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee\ (\neg p \wedge q \wedge r) \ \vee\ (q \wedge \bot)\\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee\ (\neg p \wedge q \wedge r) \ \vee\ \bot\\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee\ (\neg p \wedge q \wedge r)
\end{aligned}
$$

Normal Forms of Fomulas

It is easy to construct a formula in CNF or DNF for a given truth table:

p	q	r	φ
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

DNF:
$(\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r)$

CNF:
$(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

Normal Forms of Fomulas

When we consider a fixed **finite** set of atomic propositions At:

- **Complete disjunctive normal form** (**CDNF**) — a formula in DNF, where every elementary conjunction contains every atomic proposition from At exactly once.

  Example: $(p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)$

- **Complete conjunctive normal form** (**CCNF**) — a formula in CNF, where every clause contains every atomic proposition from At exactly once.

  Example: $(p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$

Remark: In the examples is $At = \{p, q, r\}$.

Minimal Sets of Logical Connectives

We can see from the previous discussion that connectives "$\neg$", "$\wedge$", and "$\vee$" suffice for contructing a formula for every truth table.

In fact, some smaller sets of logical connectives are sufficient for this purpose:

- "$\neg$", "$\wedge$":
  $\varphi \vee \psi$ can be expressed as $\neg(\neg\varphi \wedge \neg\psi)$

- "$\neg$", "$\vee$":
  $\varphi \wedge \psi$ can be expressed as $\neg(\neg\varphi \vee \neg\psi)$

- "$\neg$", "$\rightarrow$":
  $\varphi \vee \psi$ can be expressed as $\neg\varphi \rightarrow \psi$
  $\varphi \wedge \psi$ can be expressed as $\neg(\varphi \rightarrow \neg\psi)$

Minimal Sets of Logical Connectives

- "$\rightarrow$", "$\bot$":
  $\neg\varphi$ can be expressed as $\varphi \rightarrow \bot$
  $\varphi \vee \psi$ can be expressed as $(\varphi \rightarrow \bot) \rightarrow \psi$
  $\varphi \wedge \psi$ can be expressed as $(\varphi \rightarrow (\psi \rightarrow \bot)) \rightarrow \bot$

- "$|$" — NAND — Sheffer stroke (also denoted by "$\uparrow$"):

φ	ψ	$\varphi \mid \psi$
0	0	1
0	1	1
1	0	1
1	1	0

  $\neg\varphi$ can be expressed as $\varphi \mid \varphi$
  $\varphi \vee \psi$ can be expressed as $(\varphi \mid \varphi) \mid (\psi \mid \psi)$
  $\varphi \wedge \psi$ can be expressed as $(\varphi \mid \psi) \mid (\varphi \mid \psi)$

Minimal Sets of Logical Connectives

- "$\downarrow$" — NOR — Peirce's arrow:

φ	ψ	$\varphi \downarrow \psi$
0	0	1
0	1	0
1	0	0
1	1	0

$\neg\varphi$ can be expressed as $\varphi \downarrow \varphi$
$\varphi \vee \psi$ can be expressed as $(\varphi \downarrow \psi) \downarrow (\varphi \downarrow \psi)$
$\varphi \wedge \psi$ can be expressed as $(\varphi \downarrow \varphi) \downarrow (\psi \downarrow \psi)$

Logical Entailment

Definition

Formula ψ **logically follows** from formula φ if in every truth valuation v, where φ is true, ψ is true as well.

The fact that ψ logically follows from φ is denoted

$$\varphi \models \psi.$$

- φ — assumption
- ψ — conclusion

Formula ψ logically follows from φ (i.e., $\varphi \models \psi$) iff $\varphi \rightarrow \psi$ is a tautology.

Logical Entailment

Example: Formula $r \rightarrow p$ logically follows from fomula $p \vee (q \wedge \neg r)$, i.e.,

$$p \vee (q \wedge \neg r) \models r \rightarrow p$$

	p	q	r	$p \vee (q \wedge \neg r)$	$r \rightarrow p$
	0	0	0	0	1
	0	0	1	0	0
$*$	0	1	0	1	1
	0	1	1	0	0
$*$	1	0	0	1	1
$*$	1	0	1	1	1
$*$	1	1	0	1	1
$*$	1	1	1	1	1

Logical Entailment

There can be arbitrarily many assumptions:

Definition

Formula ψ **logically follows** from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ if formula ψ is true in every truth valuation v where all these assumptions are true.

The fact the ψ logically follows from $\varphi_1, \varphi_2, \ldots, \varphi_n$ is denoted

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi.$$

- $\varphi_1, \varphi_2, \ldots, \varphi_n$ — assumptions
- ψ — conclusion

Logical Entailment

Example:

- *If the train arrives late and there are no taxis at the station, then John is late for his meeting.*
- *John is not late for his meeting.*
- *The train did arrive late.*

- *There were taxis at the station.*

$$(p \wedge \neg q) \rightarrow r,\ \neg r,\ p \ \models\ q$$

Logical Entailment

$$(p \wedge \neg q) \rightarrow r,\ \neg r,\ p \ \models\ q$$

	p	q	r	$(p \wedge \neg q) \rightarrow r$	$\neg r$	p	q
	0	0	0	1	1	0	0
	0	0	1	1	0	0	0
	0	1	0	1	1	0	1
	0	1	1	1	0	0	1
	1	0	0	0	1	1	0
	1	0	1	1	0	1	0
$*$	1	1	0	1	1	1	1
	1	1	1	1	0	1	1

Logical Entailment

For concreteness, lets assume that we have 4 assumptions. However, it is similar for any (finite) number of assumptions:

$$\varphi_1, \varphi_2, \varphi_3, \varphi_4 \models \psi$$

iff

$$\varphi_1, \varphi_2, \varphi_3 \models \varphi_4 \rightarrow \psi$$

iff

$$\varphi_1, \varphi_2 \models \varphi_3 \rightarrow (\varphi_4 \rightarrow \psi)$$

iff

$$\varphi_1 \models \varphi_2 \rightarrow (\varphi_3 \rightarrow (\varphi_4 \rightarrow \psi))$$

iff

$$\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\varphi_3 \rightarrow (\varphi_4 \rightarrow \psi)))$$

(i.e., iff $\varphi_1 \rightarrow (\varphi_2 \rightarrow (\varphi_3 \rightarrow (\varphi_4 \rightarrow \psi)))$ is a tautology)

Logical Entailment

Example: It really holds that

$$(p \wedge \neg q) \rightarrow r,\ \neg r,\ p\ \models\ q$$

(i.e., the conclusion q logically follows from assumption $(p \wedge \neg q) \rightarrow r$, $\neg r$ and p), since

$$((p \wedge \neg q) \rightarrow r) \rightarrow (\neg r \rightarrow (p \rightarrow q))$$

is a tautology.

(This can be verified by the table method or by finding a semantic contradiction.)

Logical Entailment

Deduction theorem

$$\varphi_1, \varphi_2, \ldots \varphi_n, \chi \models \psi$$

iff

$$\varphi_1, \varphi_2, \ldots \varphi_n \models \chi \rightarrow \psi$$

Logical Entailment

$$\varphi_1, \varphi_2, \ldots \varphi_n \models \psi$$

iff

$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n) \rightarrow \psi$ is a tautology

Justification (for the case with four assumptions):

$$\begin{aligned}
& \varphi_1 \rightarrow (\varphi_2 \rightarrow (\varphi_3 \rightarrow (\varphi_4 \rightarrow \psi))) \\
\Leftrightarrow\ & \varphi_1 \rightarrow (\varphi_2 \rightarrow ((\varphi_3 \wedge \varphi_4) \rightarrow \psi)) \\
\Leftrightarrow\ & \varphi_1 \rightarrow ((\varphi_2 \wedge \varphi_3 \wedge \varphi_4) \rightarrow \psi)) \\
\Leftrightarrow\ & (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4) \rightarrow \psi
\end{aligned}$$

Remark: It uses the equivalence $p \rightarrow (q \rightarrow r) \ \Leftrightarrow\ (p \wedge q) \rightarrow r$.

Logical Entailment

$$\varphi_1, \varphi_2, \ldots \varphi_n \models \psi$$

iff

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \quad \Leftrightarrow \quad \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \psi$$

$$\varphi \;\Leftrightarrow\; \psi$$

iff

$\varphi \models \psi$ and $\psi \models \varphi$

Logical Entailment

If a formula ψ is a **tautology** then it logically follows from every set of assumptions, i.e., for every set of assumptions $\varphi_1, \varphi_2, \ldots \varphi_n$ it holds that

$$\varphi_1, \varphi_2, \ldots \varphi_n \models \psi$$

In particular, if ψ is a tautology then it follows from the empty set of assumptions:

$$\models \psi$$

Tautologies are the only formulas that logically follow from the empty set of assumptions.

Logical Entailment

Let us assume that

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \chi_1 \qquad\qquad \varphi_1, \varphi_2, \ldots, \varphi_n \models \chi_2$$

and also $\chi_1, \chi_2 \models \psi$.

Then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.

Example:

- If $\varphi_1, \varphi_2, \varphi_3 \models (q \vee \neg p)$ and $\varphi_1, \varphi_2, \varphi_3 \models \neg s$ then

  $$\varphi_1, \varphi_2, \varphi_3 \models (q \vee \neg p) \wedge \neg s$$

  because $q \vee \neg p, \neg s \models (q \vee \neg p) \wedge \neg s$.

Logical Entailment

Example:

- If $\varphi_1, \varphi_2, \varphi_3 \models p \rightarrow q$ and $\varphi_1, \varphi_2, \varphi_3 \models p$ then

  $$\varphi_1, \varphi_2, \varphi_3 \models q$$

  because $p \rightarrow q,\ p \models q$.

- If $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \models \neg p \rightarrow \neg q$ then

  $$\varphi_1, \varphi_2, \varphi_3, \varphi_4 \models q \rightarrow p$$

  because $\neg p \rightarrow \neg p \models q \rightarrow p$.

Logical Entailment

When we try to prove that ψ logically follows from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, we can proceed via smaller steps.

We start we the assumptions, for example:

$$\varphi_1,\ \varphi_2,\ \varphi_3$$

Then we gradually add other formulas in such a way that every newly added formula logically follows from the previous formulas. For example:

$$\varphi_1,\ \varphi_2,\ \varphi_3,\ \chi_1,\ \chi_2,\ \chi_3,\ \chi_4,\ \chi_5,\ \chi_6,\ \chi_7,\ \chi_8,\ \psi$$

Logical Entailment

Assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are **inconsistent** (**contradictory**) if there is no truth valuation v, in which all these assumptions would be true.

Assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are inconsistent iff

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$$

is a contradiction.

Example: Assumptions $p \rightarrow q$, $r \rightarrow p$, r, $\neg q$ are inconsistent.

Logical Entailment

In the case when assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are inconsistent, $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$ is a contradition and so

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \ \Leftrightarrow \ \bot.$$

In this case we have

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \quad \Leftrightarrow \quad \bot \quad \Leftrightarrow \quad \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \bot,$$

from which follows that

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \bot.$$

Logical Entailment

Assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are inconsistent iff

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \bot .$$

Because $\bot \rightarrow \psi$ is a tautology (for any formula ψ), it holds for every ψ that if assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are inconsistent then

$$\varphi_1, \varphi_2, \ldots, \varphi_n \ \models \ \psi .$$

So every formula logically follows from inconsistent assumptions.

In particular, it holds for every formula χ that both χ and $\neg\chi$ logically follow from inconststent assumptions.

Logical Entailment

Assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are inconsistent iff there is a formula χ such that

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \chi \qquad \text{and} \qquad \varphi_1, \varphi_2, \ldots, \varphi_n \models \neg\chi$$

Formula $\chi \rightarrow (\neg\chi \rightarrow \bot)$ is a tautology (for each formula χ).

So if

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \chi \qquad \text{and} \qquad \varphi_1, \varphi_2, \ldots, \varphi_n \models \neg\chi$$

then also

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \bot.$$

On the other hand, if $\bot$ is true under the given assumptions then also χ and $\neg\chi$ are true under these assumptions.

Logical Entailment

The principle of proof by contradiction

$$\varphi_1, \varphi_2, \ldots \varphi_n \models \psi$$

iff

assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n, \neg\psi$ are inconsistent

Assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n, \neg\psi$ are inconsistent iff

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \neg\psi \models \bot,$$

which holds iff

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \neg\psi \rightarrow \bot.$$

It holds that $\neg\psi \rightarrow \bot \Leftrightarrow \psi$ (since $\neg p \rightarrow \bot \Leftrightarrow \neg\neg p \vee \bot \Leftrightarrow \neg\neg p \Leftrightarrow p$).

Resolution Method

The **resolution method** is one of algorithms for finding out whether a given conclusion follows from given assumptions.

It solves the following problem:

Input: Formulas φ_1, φ_2, ..., φ_n, ψ.

Question: Is it true that $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$?

Remark: The method can be used for finding out whether a given formula is a tautology, a contradiction, or satisfiable.

Different variants of the resolution method are used for example in some systems for automatic theorem proving and also in implementations of logic programming languages such as Prolog.

Resolution Method

- It works with formulas in CNF.
- It constructs a proof that the given conclusion follows from the assumptions.
- It is a proof by contradiction — the algorithm generates successively formulas following from the assumptions

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \neg\psi$$

- A computation can finish in two different ways:
  - A contradiction is found, i.e., formula $\perp$ is derived — then the conclusion ψ logically follows from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$.
  - The algorithm does not succeed in deriving formula $\perp$ and no other new formulas can be added — then the conclusion ψ does not follow from the assumptions.

Resolution Method

- The resolution method works with formulas that have the form of elementary disjunctions, e.g.,

  $$(\neg p \vee q \vee \neg s \vee \neg t)$$

  These formulas are called **clauses**.

- A special case of clause is the **empty clause** $\perp$ that represents a found contradiction.

- The algorithm starts the computation by transforming formulas

  $$\varphi_1, \varphi_2, \ldots, \varphi_n, \neg\psi$$

  to CNF. Then it takes all clauses from the transformed formulas as the initial set of assumptions

  $$\chi_1, \chi_2, \ldots, \chi_m.$$

Resolution Method

For generating other clauses, which are added to already constructed clauses, the algorithm uses so called **resolution rule** (or **resolution principle**):

For each formulas φ, ψ and χ it holds that

$$\varphi \vee \psi, \neg\varphi \vee \chi \models \psi \vee \chi$$

In the resolution method, this principle is used only for clauses.
In the resolution method, $\varphi \vee \psi$, $\neg\varphi \vee \chi$ and $\psi \vee \chi$ are always clauses, and φ is always an atomic proposition.

Example: From clauses

$$p \vee \neg q \vee r \vee s \qquad \text{a} \qquad \neg r \vee t \vee \neg u$$

we can derive the following clause by the resolution rule:
$p \vee \neg q \vee s \vee t \vee \neg u$.

Resolution Method

Remarks:

- An order of literals in a clause is not important.
- Multiple occurrences of the same literals in one clause can be eliminated.
- If a currently generated clause is the same as some previously generated clause (and differs only in the order of literals), it makes no sense to add it.
- Clauses containing both literals p and $\neg p$ are equivalent to $\top$ and can be eliminated.
- Clauses can be used for the application of the resolution rule repeatedly (with other clauses).

Resolution Method

Some special cases of the use of the resolution rule:

- One of clauses contains just one literal and the other more than one literal:

  From clauses

  $$\neg q \qquad\qquad p \vee q \vee \neg t$$

  we can derive clause $p \vee \neg t$.

- Both clauses contain just one literal:

  From clauses

  $$p \qquad\qquad \neg p$$

  we can derive the empty clause $\perp$, i.e., the contradiction.

Resolution Method

We want to check validity of the following deduction:

- *It is not true that Jane is at school and Peter is not at home.*
- *Jane is not at school or it's a working day or it's raining.*
- *If it's a working day then Peter is not at home.*

- *If Jane is at school then it's raining.*

At first, we formalize the individual propositions by formulae of propositional logic:

$$\begin{array}{l} \neg(j \wedge \neg p) \\ \neg j \vee d \vee r \\ d \rightarrow \neg p \\ \hline j \rightarrow r \end{array}$$

j – Jane is at school

p – Peter is at home

d – it's a working day

r – it's raining

Resolution Method

$$\begin{array}{l} \neg(j \wedge \neg p) \\ \neg j \vee d \vee r \\ d \rightarrow \neg p \\ \hline j \rightarrow r \end{array}$$

We transform the individual assumptions into CNF:

- $\neg(j \wedge \neg p) \;\Leftrightarrow\; \neg j \vee p$
- $\neg j \vee d \vee r$
- $d \rightarrow \neg p \;\Leftrightarrow\; \neg d \vee \neg p$

We negate the conclusion and transform it into CNF:

- $\neg(j \rightarrow r) \;\Leftrightarrow\; j \wedge \neg r$

Resolution Method

Let us write down the individual clauses:

1. $\neg j \vee p$ – assumption 1
2. $\neg j \vee d \vee r$ – assumption 2
3. $\neg d \vee \neg p$ – assumption 3
4. j – clause 1 of the negated conclusion
5. $\neg r$ – clause 2 of the negated conclusion

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4
7.	$d \vee r$	– resolution: 2,4

Resolution Method

Let us write down the individual clauses:

1. $\neg j \lor p$ – assumption 1
2. $\neg j \lor d \lor r$ – assumption 2
3. $\neg d \lor \neg p$ – assumption 3
4. j – clause 1 of the negated conclusion
5. $\neg r$ – clause 2 of the negated conclusion

---

6. p – resolution: 1,4
7. $d \lor r$ – resolution: 2,4
8. $\neg d$ – resolution: 3,6

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4
7.	$d \vee r$	– resolution: 2,4
8.	$\neg d$	– resolution: 3,6
9.	r	– resolution: 7,8

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4
7.	$d \vee r$	– resolution: 2,4
8.	$\neg d$	– resolution: 3,6
9.	r	– resolution: 7,8
10.	$\bot$	– resolution: 5,9

A contradition was derived, so the conclusion really follows from the given assumptions.

Resolution Method

Remarks:

- The resolution method can be viewed as a construction of one “big” formula in CNF, which is equivalent to
$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \neg\psi,$$
and which is constructed by a successive addition of clauses.

- If a contradiction can not be generated, then the derived clauses can be used for finding a truth valuation v where the assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are true and the conclusion ψ is false.

Resolution Method

- It is also possible to proceed by a direct method, where the algorithm starts only with assumptions
$$\varphi_1, \varphi_2, \ldots, \varphi_n$$
and tries to generate all clauses of the conclusion ψ.

  In this approach, it is not guaranteed that the algorithm succeeds in all cases when a conclusion ψ logically follows from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$.

  Example: Clause $p \vee q$ cannot be generated this way from the assumption p, although it holds that
$$p \models p \vee q.$$

Predicate Logic

Predicate Logic

- *Fish are vertebrates living in water.*
- *Carps are fish.*
- *There exists at least one carp.*

- *There exists at least one vertebrate living in water.*

- *Triangles are convex polygons.*
- *Equilateral triangles are triangles.*
- *There exists at least one equilateral triangle.*

- *There exists at least one convex polygon.*

Predicate Logic

- *Fish are vertebrates living in water.*
- *Carps are fish.*
- *There exists at least one carp.*

- *There exists at least one vertebrate living in water.*

The use of **variables**:

- *For each* x *it holds that if* x *is a fish then* x *is a vertebrate and* x *lives in water.*
- *For each* x *it holds that if* x *is a carp then* x *is a fish.*
- *There exists at least one* x *such that* x *is a carp.*

- *There exists at least one* x *such that* x *is a vertebrate and* x *lives in water.*

Predicate Logic

- *Triangles are convex polygons.*
- *Equilateral triangles are triangles.*
- *There exists at least one equilateral triangle.*

- *There exists at least one convex polygon.*

The use of **variables**:

- *For each x it holds that if x is a triangle then x is a polygon and x is convex.*
- *For each x it holds that if x is an equilateral triangle then x is a triangle.*
- *There exists at least one x such that x is an equilateral triangle.*

- *There exists at least one x such that x is a polygon and x is convex.*

Predicate Logic

- *For each x it holds that if x has property P then x has property Q and x has property R.*
- *For each x it holds that if x has property S then x has property P.*
- *There exists at least one x such that x has property S.*

- *There exists at least one x such that x has property Q and x has property R.*

P	is a fish	is a triangle
Q	is a vertebrate	is a polygon
R	lives in water	is convex
S	is a carp	is an equilateral triangle

Predicate Logic

- *For each x it holds that if $P(x)$ then $Q(x)$ and $R(x)$.*
- *For each x it holds that if $S(x)$ then $P(x)$.*
- *There exists x such that $S(x)$.*

- *There exists x such that $Q(x)$ and $R(x)$.*

$P(x)$	x is a fish	x is a triangle
$Q(x)$	x is a vertebrate	x is a polygon
$R(x)$	x lives in water	x is convex
$S(x)$	x is a carp	x is an equilateral triangle

Predicate Logic

- *For each* x, $(P(x) \rightarrow (Q(x) \wedge R(x)))$.
- *For each* x, $(S(x) \rightarrow P(x))$.
- *There exists* x *such that* $S(x)$.

- *There exists* x *such that* $(Q(x) \wedge R(x))$.

$P(x)$	x is a fish	x is a triangle
$Q(x)$	x is a vertebrate	x is a polygon
$R(x)$	x lives in water	x is convex
$S(x)$	x is a carp	x is an equilateral triangle

Predicate Logic

- $\forall x(P(x) \rightarrow (Q(x) \wedge R(x)))$
- $\forall x(S(x) \rightarrow P(x))$
- $\exists x\, S(x)$

- $\exists x(Q(x) \wedge R(x))$

$P(x)$	x is a fish	x is a triangle
$Q(x)$	x is a vertebrate	x is a polygon
$R(x)$	x lives in water	x is convex
$S(x)$	x is a carp	x is an equilateral triangle

- $\forall$ — universal quantifier (*"for all"*)
- $\exists$ — existential quantifier (*"there exists"*)

Predicate Logic

Formulas of propositional logic express propositions about objects with some properties and which can be in some relationships.

Interpretation or **interpretation structure** — a particular set of these objects, their properties and relationships.

Universe — the set of all objects in a given interpretation

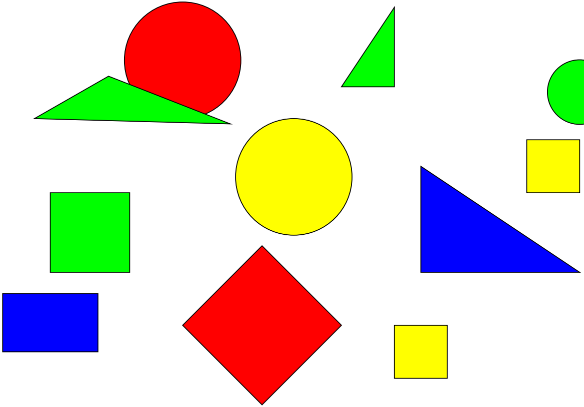
- An arbitrary **non-empty** set can be the universe.
- Objects in a given universe are called the **elements** of the universe.

Valuation — an assignment of elements of the universe to variables

The truth of formulas depends on a given interpretation and valuation.

Predicate Logic

An example of a universe:



Predicate Logic

Other examples of universes:

- Some precisely specified set of people, for example, the set of people that live in some specified house (*"John Smith"*, *"John Doe"*, ...)
- The set of all books in a given library.
- The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- The set of all points in a plane.
- The set $\{a, b, c, d, e\}$.
- The set $\{a\}$.

Variables

Variables — x, y, z, ..., possibly with indexes — x_0, x_1, x_2, ...

It is assumed that there are infinitely many variables.

Valuation — an assignment of elements of the universe to the variables

Example:

- Universe — a set of people; valuation v, where:

  $v(x) = $ *"John Doe"*
  $v(y) = $ *"Mary Smith"*
  ...

- Universe — the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$; valuation v, where

$$v(x) = 57 \qquad v(y) = 3 \qquad v(z) = 57 \qquad \ldots$$

Predicates

Predicates — P, Q, R, ...

- **Unary predicates** — they represent **properties** of elements of the universe

  Example: Predicate P representing the property *"to be blue"*:

  $$P(x) \quad — \quad \text{“}x \textit{ is blue”}$$

  A unary predicate assigns truth values to the elements of the universe.

  E.g., the value of $P(x)$ can be:

  - 1 — the element assigned to variable x has property P (i.e., it is blue)
  - 0 — the element assigned to variable x does not have this property P (i.e., it is not blue)

Predicates

- **Binary predicates** — they represent **relationships** between pairs of elements of the universe

  Example: Predicate R representing the relationship *"to be a parent of"*:

  $$R(x, y) \quad — \quad \text{"}x \text{ is a parent of } y\text{"}$$

  A binary predicate assigns truth values to pair of elements of the universe.

  E.g., the value of $R(x, y)$ can be:

  - 1 — when x and y are in the given relationship (i.e., when x is a parent of y)
  - 0 — when x and y are not in the given relationship (i.e., when x is not a parent of y)

Predicates

We can consider predicates of arbitrary arities.

For example:

- **Ternary** predicate T (i.e., predicate of arity 3) representing the relationship between parents and their child:

  $$T(x, y, z)$$

  — x and y are parents of child z, and x is his/her mother and y is his/her father

- **Nulary** predicates (i.e., precates of arity 0) can be viewed as atomic propositions, not related to the elements of the universe.

Formulas of Predicate Logic

Atomic formula — a predicate applied on some variables

Example:

- P — a unary predicate representing property *"to be blue"*
- Q — a unary predicate representing propery *"to be a square"*
- R — a binary predicate representing relationships *"overlaps"*

$P(x)$	—	*"x is blue"*
$P(y)$	—	*"y is blue"*
$Q(y)$	—	*"y is a square"*
$R(z,x)$	—	*"z overlaps x"*
$R(y,y)$	—	*"y overlaps itself"*

Remark: Later, we will extend the notion of an atomic formula a little bit.

Formulas of Predicate Logic

Using **logical connectives** ("$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$"), more complicated formulas can be created from simpler formulas, similarly as in propositional logic.

Example:

- P — unary predicate representing property *"is blue"*
- Q — unary predicate representing property *"is a square"*
- R — binary predicate representing relationship *"overlaps"*

"If x is a blue square or y does not overlap x, then z is not a square."

$$\big((P(x) \wedge Q(x)) \vee \neg R(y,x)\big) \rightarrow \neg Q(z)$$

Formulas of Predicate Logic

Using **logical connectives** ("$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$"), more complicated formulas can be created from simpler formulas, similarly as in propositional logic.

Example:

- P — unary predicate representing property *"is a woman"*
- Q — unary predicate representing property *"has dark hair"*
- R — binary predicate representing relationship *"is a parent of"*

"If x is a woman with dark hair or y is not a parent of x, then z does not have dark hair."

$$\big((P(x) \wedge Q(x)) \vee \neg R(y,x)\big) \rightarrow \neg Q(z)$$

Formulas of Predicate Logic

Using **logical connectives** ("$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$"), more complicated formulas can be created from simpler formulas, similarly as in propositional logic.

Example:

- P — unary predicate representing property *"is even"*
- Q — unary predicate representing property *"is a prime"*
- R — binary predicate representing relationship *"is greater than"*

"If x is an even prime or y is not greater than x, then z is not a prime."

$$\big((P(x) \wedge Q(x)) \vee \neg R(y,x)\big) \rightarrow \neg Q(z)$$

Quantifiers

Universal quantifier — symbol "$\forall$"

If φ is a formula representing some proposition then

$$\forall x \varphi$$

is a formula representing proposition

"for every x φ holds".

Example: P — *"to be a square"*

$$\forall x\, P(x)$$

- *"For every x it holds that x is a square."*
- *"Every x is a square."*
- *"All elements are squares."*

Quantifiers

Example:

- *"For every* x *it holds that if* x *is a square then* x *is green."*
- *"For each* x *it holds that if* x *is a square then* x *is green."*
- *"For all* x *it holds that if* x *is a square then* x *is green."*
- *"All squares are green."*

$$\forall x(P(x) \rightarrow Q(x))$$

- P — *"to be a square"* (arity 1)
- Q — *"to be green"* (arity 1)

Quantifiers

Example:

- *"If it holds for all* x *that* x *is a square or* x *is green then it holds for all* y *that* y *is a triangle."*
- *"If every object is a square or is green then all objects are triangles."*

$$\forall x(P(x) \vee Q(x)) \rightarrow \forall y T(y)$$

- P — *"to be a square"* (arity 1)
- Q — *"to be green"* (arity 1)
- T — *"to be a triangle"* (arity 1)

Quantifiers

There is a big difference between the following formulas:

- $P(x)$ — *"x is a square"*

  It claims something about **one** particular element assigned to variable x.

  The truth of this claim depends on the particular element assigned to variable x, i.e., on the particular valuation.

- $\forall x P(x)$ — *"every x is a square"* (i.e., *"all elements are squares"*)

  It claims something about **all** elements of the universe.

  The truth of this claim does not depend on a valuation.

Quantifiers

Example:

- *"If* x *is a prime then* x *is odd."*

$$P(x) \rightarrow L(x)$$

- *"For every* x *it holds that if* x *is a prime then it is odd"*. (I.e., *"all primes are odd"*.)

$$\forall x(P(x) \rightarrow L(x))$$

Predicates:

- P — *"to be a prime"* (arity 1)
- L — *"to be odd"* (arity 1)

Quantifiers

Example:

- *"It holds for every y that if y is green then x overlaps y."*
- *"Object x overlaps all green objects."*

$$\forall y(G(y) \rightarrow R(x,y))$$

Predicates:

- R — *"overlaps"* (arity 2)
- G — *"to be green"* (arity 1)

Quantifiers

Example:

- *"It holds for every x that it holds for every y that if x is a parent of y then x loves y."*
- *"It holds for each x and y that if x is a parent of y then x loves y."*
- *"For every pair of elements x and y it holds that if x is a parent of y then x loves y."*

$$\forall x \forall y (R(x,y) \rightarrow S(x,y))$$

Predicates:

- R — *"is a parent"* (arity 2)
- S — *"loves"* (arity 2)

Quantifiers

Existential quantifier — symbol “$\exists$”

If φ is a formula representing some proposition then

$$\exists x \varphi$$

is a formula representing proposition

“there exists x, for which φ holds”.

Example: P — *“to be a square”*

$$\exists x\, P(x)$$

- *“There exists x, for which it holds that x is a square.”*
- *“There is x such that x is a square.”*
- *“There exists at least one square.”*

Quantifiers

Example:

- *“There exists* x*, for which it holds that* x *is a square and* x *is green.”*
- *“There is* x *such that* x *is a square and* x *is green.”*
- *“For some* x *it holds that* x *is a square and* x *is green.”*
- *“There exists a green square.”*
- *“Some squares are green.”*
- *“At least one* x *is a green square.”*

$$\exists x(P(x) \land Q(x))$$

Predicates:

- P — *“to be a square”* (arity 1)
- Q — *“to be green”* (arity 1)

Quantifiers

Example:

- *“There exists* x *such that for each* y *it holds that* x *is greater than* y*.”*

$$\exists x \forall y\, P(x, y)$$

- *“For each* y *there is* x *such that* x *is greater than* y*.”*

$$\forall y \exists x\, P(x, y)$$

P — *“to be greater than”* (arity 2)

Syntax of Formulas of Predicate Logic

Alphabet:

- **logical connectives** — "$\neg$", "$\wedge$", "$\vee$" "$\rightarrow$", "$\leftrightarrow$"
- **quantifiers** — "$\forall$", "$\exists$"
- **auxiliary symbols** — "(", ")", ","
- **variables** — "x", "y", "z", . . . , "x_0", "x_1", "x_2", . . .
- **predicate symbols** — for example symbols "P", "Q", "R", etc. (for each symbol, its arity must be specified)
- . . .

Remark: Other types of symbols will be described later.

Syntax of Formulas of Predicate Logic

Definition

Well-formed **atomic formulas** of predicate logic are formulas of the form:

- $P(x_1, x_2, \ldots, x_n)$, where P is a predicate symbol of arity n and $x_1, x_2, \ldots, x_n$ are (not necessarily different) variables.
- $\ldots$

Remark: This is not the whole definition. Later, it will be generalized a little bit and some additional items will be added.

Example:

$$P(x, y) \qquad R(z, z, z) \qquad S(y)$$

Syntax of Formulas of Predicate Logic

Definition

Well-formed **formulas of predicate logic** are sequences of symbols constructed according to the following rules:

1. Well-formed atomic formulas are well-formed formulas.
2. If φ and ψ are well-formed formulas, then also $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ a $(\varphi \leftrightarrow \psi)$ are well-formed formulas.
3. If φ is a well-formed formula and x is a variable, then $\forall x \varphi$ and $\exists x \varphi$ are well-formed formulas.
4. There are no other well-formed formulas than those constructed according to the previous rules.

Syntax of Formulas of Predicate Logic

Notions like

- subformulas
- an abstract syntax tree

are introduced in a similar way like in propositional logic (they are only extended with the additional constructions not present in propositional logic).

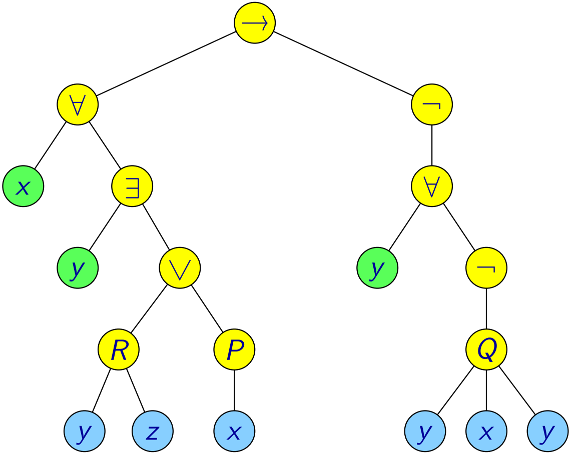
Convention for omitting parentheses:

- The same conventions as in propositional logic.
- Quantifiers ("$\forall$" and "$\exists$") have the same priority as negation ("$\neg$"), i.e., the highest priority.

Syntax of Formulas of Predicate Logic

An abstract syntax tree of formula

$$\forall x \exists y (R(y,z) \vee P(x)) \rightarrow \neg \forall y \neg Q(y,x,y)$$



Free and Bound Occurrences of Variables

Every occurrence of variable x in a subformula of the form $\exists x \varphi$ or $\forall x \varphi$ is **bound**.

An occurrence of a variable, which is not bound, is **free**.

Example: Formula

$$\forall x \exists y (R(y,z) \vee P(x)) \rightarrow \neg \forall y \neg Q(y,x,y)$$

- y in subformula $R(y,z)$ — the bound occurrence ($\exists y$)
- z in subformula $R(y,z)$ — the free occurrence
- x in subformula $P(x)$ — the bound occurrence ($\forall x$)
- both occurrences of y in subformula $Q(y,x,y)$ — the bound occurrences ($\forall y$)
- x in subformula $Q(y,x,y)$ — the free occurrence

Free and Bound Occurrences of Variables

The set of those variables, which occur as **free** variables in formula φ, will be denoted $\mathit{free}(\varphi)$.

Example:

- If φ is formula $P(x,y)$, then $\mathit{free}(\varphi) = \{x,y\}$.
- If ψ is formula $\exists x \exists y P(x,y)$, then $\mathit{free}(\psi) = \emptyset$.
- If χ is formula
$$\forall x \exists y (R(y,z) \vee P(x)) \to \neg \forall y \neg Q(y,x,y)$$
then $\mathit{free}(\chi) = \{x,z\}$.

Free and Bound Occurrences of Variables

The set of free variables $\mathit{free}(\varphi)$ can be described by the following inductive definition:

- $\mathit{free}(P(x_1, x_2, \ldots, x_n)) = \{x_1, x_2, \ldots, x_n\}$
  (where P is a predicate symbol)
- $\mathit{free}(\neg\varphi) = \mathit{free}(\varphi)$
- $\mathit{free}(\varphi \wedge \psi) = \mathit{free}(\varphi) \cup \mathit{free}(\psi)$
  (it is similar for formulas of the form $\varphi \vee \psi$, $\varphi \rightarrow \psi$, and $\varphi \leftrightarrow \psi$)
- $\mathit{free}(\forall x \varphi) = \mathit{free}(\varphi) - \{x\}$ (where x is a variable)
- $\mathit{free}(\exists x \varphi) = \mathit{free}(\varphi) - \{x\}$ (where x is a variable)

Semantics of Predicate Logic

Formulas are evaluated in a given **interpretation** (**interpretation structure**) and **valuation**.

The fact that formula φ is true (i.e., it has truth value 1) in interpretation $\mathcal{A}$ and valuation v, is denoted

$$\mathcal{A}, v \models \varphi$$

The fact that formula φ is false (has truth value 0) in interpretation $\mathcal{A}$ and valuation v, is denoted $\mathcal{A}, v \not\models \varphi$.

Semantics of Predicate Logic

An **interpretation** $\mathcal{A}$ is a structure consisting of the following items:

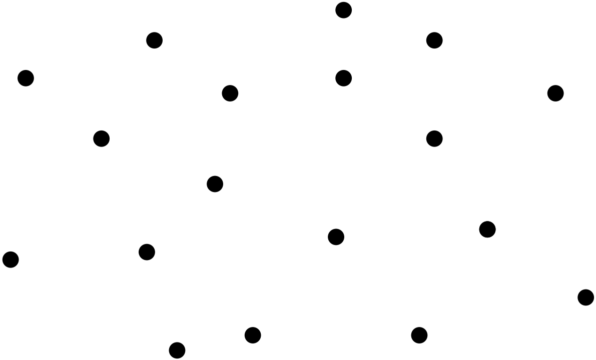
- **Universe** A — an arbitraty non-empty set
- Some subset of the set A is assigned to every unary predicate symbol P — it is denoted $P^{\mathcal{A}}$.
  (And so $P^{\mathcal{A}} \subseteq A$.)
- Some binary relation on A is assigned to every binary predicate symbol Q — it is denoted $Q^{\mathcal{A}}$.
  (And so $Q^{\mathcal{A}} \subseteq A \times A$.)
- It is similar for predicate symbols with other arities (3, 4, 5, . . .).

Remark: This definition is not complete yet, and it will be later extended by other items.

Semantics of Predicate Logic

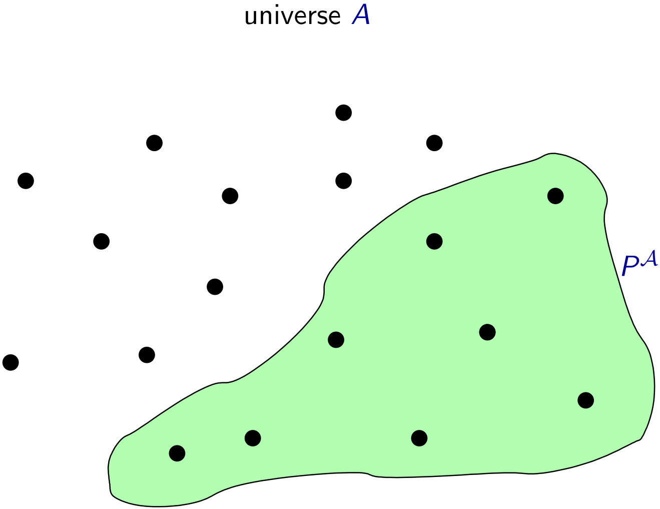
An example of an interpretation $\mathcal{A}$:

universe A



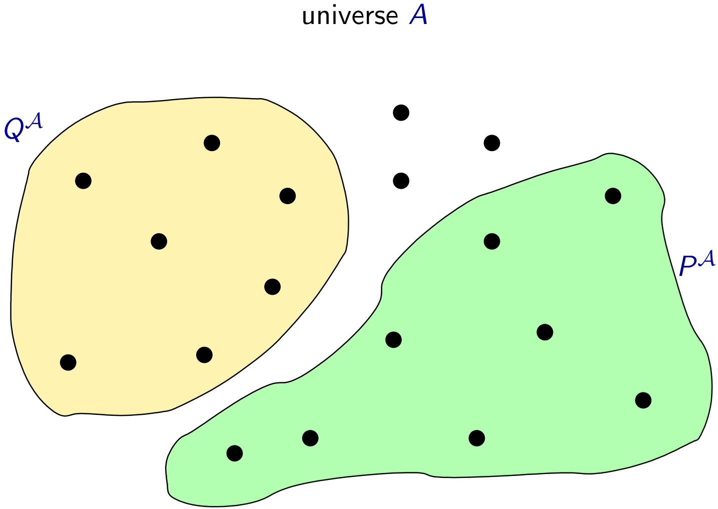
Semantics of Predicate Logic

An example of an interpretation $\mathcal{A}$:



Semantics of Predicate Logic

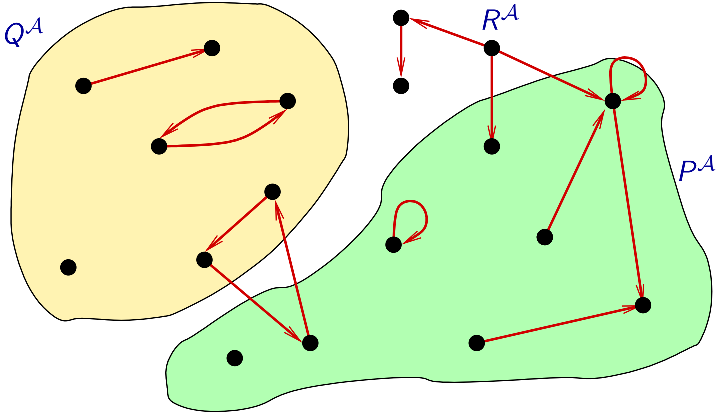
An example of an interpretation $\mathcal{A}$:



Semantics of Predicate Logic

An example of an interpretation $\mathcal{A}$:

universe A



Semantics of Predicate Logic

Other example of an interpretation $\mathcal{A}$:

- universe $A = \{a, b, c, d, e, f, g\}$
- $P^{\mathcal{A}} = \{b, d, e\}$
- $Q^{\mathcal{A}} = \{a, b, e, g\}$
- $R^{\mathcal{A}} = \{(a, b),\ (a, e),\ (a, g),\ (b, b),\ (c, e),\ (f, c), (f, g),\ (g, a),\ (g, g)\}$

Semantics of Predicate Logic

Let Var be the set of all variables, i.e.,

$$Var = \{x, y, z, \dots, x_0, x_1, x_2, \dots\}$$

For a given interpretation $\mathcal{A}$ with a universe A, a **valuation** v is an arbitrary function

$$v : Var \to A$$

that assignes elements of the universe to the variables.

Remark: As we will see, in fact, only values assigned by the valuation v to variables in $free(\varphi)$ are important for determining the truth value of formula φ.

Values assigned by valuation v to the other variables are not important from this point of view.

Semantics of Predicate Logic

Let us consider an interpretation $\mathcal{A}$ with universe A and a valuation v.

Lets assume that (i.e., $x \in \mathit{Var}$) and a is an element of the universe (i.e., $a \in A$).

Notation

$$v[x \mapsto a]$$

denotes the valuation $v' : \mathit{Var} \to A$, which assignes to every variable the same value as valuation v, except that it assignes value a to variable x.

I.e., for every variable y (where $y \in \mathit{Var}$) is

$$v'(y) = \begin{cases} a & \text{if } y = x \\ v(y) & \text{otherwise} \end{cases}$$

Semantics of Predicate Logic

Example:

- universe $A = \{a, b, c, d, e, f, g, \ldots\}$

  valuation v:

  $$v(x_0) = c \qquad v(x_1) = e \qquad v(x_2) = b \qquad v(x_3) = e \qquad \ldots$$

  valuation $v[x_2 \mapsto g]$:

  $$v(x_0) = c \qquad v(x_1) = e \qquad v(x_2) = g \qquad v(x_3) = e \qquad \ldots$$

Semantics of Predicate Logic

Definition

Let us assume an interpretation $\mathcal{A}$ with universe A and a valuation v, assigning elements of the universe A to the variables.

The **truth of formulas of predicate logic** in interpretation $\mathcal{A}$ and valuation v is defined as follows:

- For a predicate P of arity n, $\mathcal{A}, v \models P(x_1, x_2, \ldots, x_n)$ iff $(v(x_1),\ v(x_2),\ \ldots\ , v(x_n)) \in P^{\mathcal{A}}$.
- $\mathcal{A}, v \models \neg\varphi$ iff $\mathcal{A}, v \not\models \varphi$.
- $\mathcal{A}, v \models \varphi \wedge \psi$ iff $\mathcal{A}, v \models \varphi$ and $\mathcal{A}, v \models \psi$.
- $\mathcal{A}, v \models \varphi \vee \psi$ iff $\mathcal{A}, v \models \varphi$ or $\mathcal{A}, v \models \psi$.
- $\mathcal{A}, v \models \varphi \rightarrow \psi$ iff $\mathcal{A}, v \not\models \varphi$ or $\mathcal{A}, v \models \psi$.
- $\mathcal{A}, v \models \varphi \leftrightarrow \psi$ iff $\mathcal{A}, v \models \varphi$ and $\mathcal{A}, v \models \psi$, or $\mathcal{A}, v \not\models \varphi$ and $\mathcal{A}, v \not\models \psi$.
- $\ldots$

Semantics of Predicate Logic

Definition (cont.)

- $\dots$
- $\mathcal{A}, v \models \forall x \varphi$ iff for **every** $a \in A$ it holds that $\mathcal{A}, v[x \mapsto a] \models \varphi$.
- $\mathcal{A}, v \models \exists x \varphi$ iff there exists some $a \in A$ such that $\mathcal{A}, v[x \mapsto a] \models \varphi$.

Remark: Those interpretations where a given formula is true are called its **models**.

Evaluation of Truth of Formulas as a Game

Let us consider a formula of the form

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \cdots \exists x_{n-1} \forall x_n \varphi ,$$

where quantifiers alternate in some arbitrary way, and where φ does not contain quantifiers.

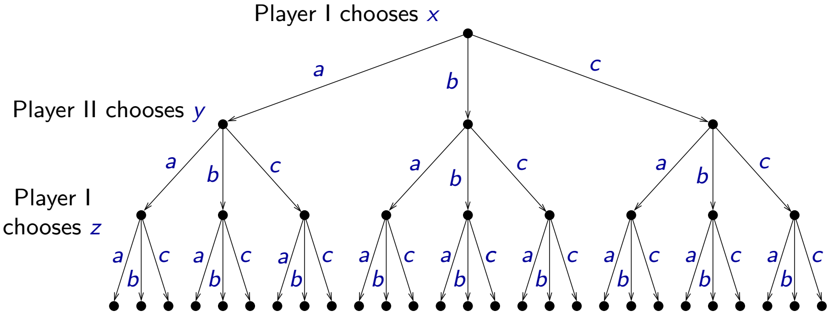
The evaluation of truth values of formulas of this form (in a given interpretation $\mathcal{A}$ and a valuation v) can be viewed as a game:

- It is played by a pair of players — **Player I** and **Player II**.
- Player I wants to show that the formula is true.
- Player II wants to show that the formula is false.
- Player I chooses values of those variables, which are bound by an existential quantifier ($\exists$).
- Player II chooses values of those variables, which are bound by an universal quantifier ($\forall$).

Evaluation of Truth of Formulas as a Game

Example: Formula $\exists x \forall y \exists z (P(x,y) \rightarrow Q(y,z))$

universe $A = \{a, b, c\}$



Evaluation of Truth of Formulas as a Game

- Formula φ is true iff Player I has a **winning strategy** in this game.
- Formula φ is false iff Player II has a winning strategy.

Strategy — determines how a player should play in every situation, i.e., it determines moves of the player for all possible moves of the other player.

Winning strategy — a strategy that guarantees a win of the given player in every play, not matter what the other player does.

Evaluation of Truth of Formulas as a Game

Example: Interpretation where universe is the set of real numbers $\mathbb{R}$ and binary predicate symbol R represents relation *"greater or equal"* (i.e., $R(x, y)$ iff $x \geq y$).

Formula $\exists x \forall y R(x, y)$ — a winning strategy of Player II:

- Player I chooses number x.
- Player II chooses number $y = x + 1$ — Player II wins since it is obviously not true that $x \geq x + 1$.

Formula $\forall y \exists x R(x, y)$ — a winning strategy of Player I:

- Player II chooses number y.
- Player I chooses number $x = y$ — Player I wins since it is obviously true that $x \geq x$.

Logically Valid Formulas

A formula φ is **logically valid** if it is true in every interpretation and valuation, i.e., if for every interpretation $\mathcal{A}$ and valuation v is

$$\mathcal{A}, v \models \varphi.$$

Example:

- $\exists x P(x) \rightarrow \exists y P(y)$
- $\forall x P(x) \land \neg \exists y Q(y) \rightarrow \forall z (P(z) \land \neg Q(z))$
- $\forall x P(x) \rightarrow \exists x P(x)$

Logically Valid Formulas

If we take an arbitrary tautology of propositional logic and replace in it all atomic propositions with arbitrary formulas of predicate logic, we obtain a logically valid formula.

Example: Tautology $p \rightarrow (q \vee p)$

- p is replaced with $\forall z(P(x,z) \leftrightarrow \neg Q(z,y))$
- q is replaced with $R(x)$

We obtain a logically valid formula

$$\forall z(P(x,z) \leftrightarrow \neg Q(z,y)) \;\rightarrow\; (R(x) \vee \forall z(P(x,z) \leftrightarrow \neg Q(z,y)))$$

Logically Equivalent Formulas

Formulas φ and ψ are **logically equivalent** if they have the same truth values in every interpretation and valuation, i.e., if for every interpretation $\mathcal{A}$ and valuation v is

$$\mathcal{A}, v \models \varphi \qquad \text{iff} \qquad \mathcal{A}, v \models \psi.$$

The fact that φ and ψ are logically equivalent is denoted

$$\varphi \Leftrightarrow \psi.$$

- Similarly as in propositional logic, we can do equivalent transformations in predicate logic.
- All equivalences that hold in propositional logic also hold in predicate logic.

Logically Equivalent Formulas

- There are other equivalences in predicate logic that have no analogy in propositional logic.

Examples of some important equivalences:

$$\begin{aligned} \neg\forall x\varphi &\Leftrightarrow \exists x\neg\varphi \\ \neg\exists x\varphi &\Leftrightarrow \forall x\neg\varphi \end{aligned}$$

$$\begin{aligned} \forall x\forall y\varphi &\Leftrightarrow \forall y\forall x\varphi \\ \exists x\exists y\varphi &\Leftrightarrow \exists y\exists x\varphi \end{aligned}$$

When $x \notin \mathit{free}(\varphi)$:

$$\begin{aligned} \forall x\varphi &\Leftrightarrow \varphi \\ \exists x\varphi &\Leftrightarrow \varphi \end{aligned}$$

Logically Equivalent Formulas

Some other important equivalences:

$$\begin{aligned} (\forall x \varphi) \wedge (\forall x \psi) &\Leftrightarrow \forall x(\varphi \wedge \psi) \\ (\exists x \varphi) \vee (\exists x \psi) &\Leftrightarrow \exists x(\varphi \vee \psi) \end{aligned}$$

When $x \notin \mathit{free}(\psi)$:

$$\begin{aligned} (\forall x \varphi) \wedge \psi &\Leftrightarrow \forall x(\varphi \wedge \psi) \\ (\forall x \varphi) \vee \psi &\Leftrightarrow \forall x(\varphi \vee \psi) \\ (\exists x \varphi) \wedge \psi &\Leftrightarrow \exists x(\varphi \wedge \psi) \\ (\exists x \varphi) \vee \psi &\Leftrightarrow \exists x(\varphi \vee \psi) \end{aligned}$$

Renaming of Bound Variables

If we rename a bound variable in a formula, we obtain an equvalent formula.

Example: $\forall x P(x,y) \Leftrightarrow \forall z P(z,y)$

- If we rename for example x to y in formula $\forall x \varphi$ or $\exists x \varphi$, the variable y **must not** occur in formula φ as a free variable.

  $\exists x P(x,y)$ is not equivalent to $\exists y P(y,y)$

- Free occurrences of variables in a subformula **must not** become bound after renaming. E.g.,

  $\exists x \forall y P(x,y)$ is not equivalent to $\exists y \forall y P(y,y)$

Substitution

Let us say that we want replace **free** occurrences of variable x with variable y (i.e., we want to substitute y for x).

This operation on formulas is called **substitution** and the resulting formula is denoted

$$\varphi[y/x].$$

Remark: In general, formulas φ and $\varphi[y/x]$ are **not** equivalent.

Example:

$$P(x,z) \qquad \text{is not equivalent to} \qquad P(y,z)$$

Renaming of Bound Variables

With the operation of substitution, the renaming of bound variables can be described by the following equivalences.

When $y \notin \mathit{free}(\forall x\varphi)$:

$$\forall x\varphi \quad \Leftrightarrow \quad \forall y(\varphi[y/x])$$

When $y \notin \mathit{free}(\exists x\varphi)$:

$$\exists x\varphi \quad \Leftrightarrow \quad \exists y(\varphi[y/x])$$

Example:

$$\exists x\forall yP(x,y) \quad \Leftrightarrow \quad \exists x\forall zP(x,z) \quad \Leftrightarrow \quad \exists y\forall zP(y,z) \quad \Leftrightarrow \quad \exists y\forall xP(y,x)$$

Logical Entailment

Definition

Conclusion ψ **logically follows** from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, which is denoted

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi,$$

if in every interpretation $\mathcal{A}$ and valuation v where assumption $\varphi_1, \varphi_2, \ldots, \varphi_n$ are true, also the conclusion ψ is true.

- All, what was said about the logical entailment in propositional logic, holds all analogously in predicate logic.

Logical Entailment

If we want to show that a given conclusion ψ **does not** follow from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, it is sufficient to find an example of one particular interpretation $\mathcal{A}$ and valuation v, where the assumptions are true and the conclusion ψ is false.

Example:

- *There exists an aquatic animal, which is meat-eating.*
- *All fish are aquatic animals.*

- *There exists a meat-eating fish.*

$$\frac{\begin{array}{c}\exists x(P(x) \wedge Q(x)) \\ \forall x(R(x) \rightarrow P(x))\end{array}}{\exists x(R(x) \wedge Q(x))}$$

$P(x)$ — *"x is an aquatic animal"*

$Q(x)$ — *"x is meat-eating"*

$R(x)$ — *"x is a fish"*

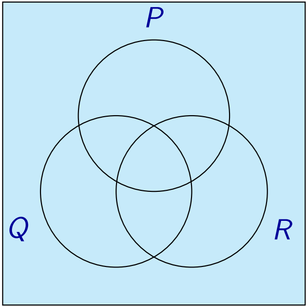
An interpretation $\mathcal{A}$ with universe $A = \{a, b\}$

$$P^{\mathcal{A}} = \{a, b\} \qquad Q^{\mathcal{A}} = \{a\} \qquad R^{\mathcal{A}} = \{b\}$$

Venn Diagrams

In general, it is difficult to find out whether a conclusion does or does not follow from given assumptions.

In cases when we have only unary predicates and there is only a small number of them (e.g., 3), we use so called **Venn diagrams** as an aid for the reasoning.



Venn Diagrams

Example:

- *Fish are vertebrates.*
- *Fish live in water.*
- *There exists at least one fish.*

- *There exists a vertebrate living in water.*

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ \forall x(P(x) \rightarrow R(x)) \\ \exists x P(x) \\ \hline \exists x(Q(x) \land R(x)) \end{array}$$

$P(x)$ — *"x is a fish"*
$Q(x)$ — *"x is a vertebrate"*
$R(x)$ — *"x lives in water"*

⟨*a solution on a whiteboard*⟩

An Example of a Proof

$$\begin{array}{l} \forall x(\neg R(x,x)) \\ \forall x \forall y \forall z(R(x,y) \wedge R(y,z) \rightarrow R(x,z)) \\ \hline \forall x \forall y(R(x,y) \rightarrow \neg R(y,x)) \end{array}$$

1.	$\forall x(\neg R(x,x))$	- assumption 1
2.	$\forall x \forall y \forall z(R(x,y) \wedge R(y,z) \rightarrow R(x,z))$	- assumption 2
3.	Lets assume arbitrary elements x and y:	
4.	Lets assume $R(x,y)$:	
5.	Lets assume $R(y,x)$:	
6.	$R(x,y) \wedge R(y,x) \rightarrow R(x,x)$	- from 2.
7.	$R(x,x)$	- from 4., 5., 6.
8.	$\neg R(x,x)$	- from 1.
9.	$\neg R(y,x)$	- contradiction of 7. and 8., so 5. does not hold
10.	$R(x,y) \rightarrow \neg R(y,x)$	- from 4., 9.
11.	$\forall x \forall y(R(x,y) \rightarrow \neg R(y,x))$	- from 3., 10.

Equality

One of the most important relations is **equality** (**identity**).

Elements x and y are equal, written

$$x = y,$$

if they are the same element.

Equality can be expressed as a predicate, e.g., we can choose that $P(x, y)$ represents proposition *"x and y are equal"*.

But $P(x, y)$ can be true in some interpretations even when x and y are distinct elements, and in some interpretations, $P(x, x)$ can be false for some element x.

Equality

Example: We would like to describe relationship *"x is a sibling of y"* using a binary predicate R, where $R(x, y)$ means *"x is a parent of y"*.

An attempt for a possible solution:

"x is a sibling of y"

iff

$$\exists z(R(z, x) \wedge R(z, y))$$

Problem: If for a given x there is an element z such that $R(z, x)$, then it is true that

$$\exists z(R(z, x) \wedge R(z, x)),$$

and so it is also true that *"x is a sibling of x"*.

Equality

Alphabet:

- . . .
- Symbol for equality: "$=$"
- . . .

Atomic formulas (cont.)

- . . .
- If x and y are variables then $x = y$ is a well-formed atomic formula.
- . . .

Symbol "$=$" is interpreted as equality in every interpretation, i.e., in every interpretation $\mathcal{A}$ and valuation v is:

- $\mathcal{A}, v \models x = y$ iff $v(x) = v(y)$.

Equality

Example: The relationship *"x is a sibling of y"* can be expressed by formula

$$\neg(x = y) \;\wedge\; \exists z(R(z,x) \wedge R(z,y)),$$

where $R(x,y)$ means that *"x is a parent of y"*.

Remark: The notation $x \neq y$ is often used instead of $\neg(x = y)$.

Equality

- *"There exists* **exactly one** x *such that* $P(x)$*"*:

$$\exists x(P(x) \land \forall y(P(y) \rightarrow x = y))$$

- *"There exist* **at least two** *elements* x *such that* $P(x)$*"*:

$$\exists x \exists y(P(x) \land P(y) \land \neg(x = y))$$

- *"There exist* **exactly two** *elements* x *such that* $P(x)$*"*:

$$\exists x \exists y(P(x) \land P(y) \land \neg(x = y) \land \forall z(P(z) \rightarrow (z = x \lor z = y)))$$

- *"There exists* **exactly one** x*, for which* φ *holds"*:

$$\exists x(\varphi \land \forall y(\varphi[y/x] \rightarrow x = y))$$

Constants

Sometimes we want to talk about some particular element of the universe.

Example: *"There exists at least one x such that John Smith is a parent of x and x is a woman."* (I.e., *"John Smith has at least one daughter."*)

If the value assigned to variable y is *"John Smith"*:

$$\exists x (R(y,x) \wedge S(x))$$

- $R(x,y)$ — *"x is a parent of y"*
- $S(x)$ — *"x is a woman"*

We could introduce an unary predicate N representing property *"to be John Smith"*:

$$\forall y (N(y) \rightarrow \exists x (R(y,x) \wedge S(x)))$$

Constants

If we have some unary predicate N where we are interested only in those interpretations where exists **exactly one** element x, for which $N(x)$ holds, it would be convenient to have some way to name this element and refer to it directly instead of using of the predicate N.

Constant symbols (**constants**) can be used for this purpose.

Alphabet:

- ...
- constant symbols: "a", "b", "c", "d", ...
- ...

Constants

In **atomic formulas**, constants can occur at the same places as variables:

$$P(c,x) \qquad Q(d) \qquad R(a,a) \qquad x = a$$

- Constants **must not** be used in quantifiers — e.g., $\exists c P(x,c)$ **is not** a well-formed formula.

Values assigned to constant symbols are determined by a given **interpretation**:

- A given interpretation $\mathcal{A}$ (with universe A) assigns to every constant symbol c some element of the universe A.

  This element is denoted $c^{\mathcal{A}}$. So $c^{\mathcal{A}} \in A$.

Constants

Example: *"There exists at least one x such that John Smith is a parent of x and x is a woman."*

$$\exists x(R(a,x) \land S(x))$$

- $R(x,y)$ — *"x is a parent of y"*
- $S(x)$ — *"x is a woman"*
- a — constant symbol representing *"John Smith"*

Constants

Example: *"Every prime is greater than one."*

$$\forall x(P(x) \rightarrow R(x, e))$$

- $P(x)$ — *"x is a prime"*
- $R(x, y)$ — *"x is greater than y"*
- e — constant symbol representing value 1

Functions

A binary relation R is a (unary) **function** if for each x there exists at most one y such that

$$(x, y) \in R.$$

This function is **total** if for each x there exists exactly one such y.

Example: Binary relation R on the set of natural numbers $\mathbb{N}$ where

$$(x, y) \in R \qquad \text{iff} \qquad y = x + 1$$

We have

$$R = \{(0,1),\ (1,2),\ (2,3),\ (3,4),\ (4,5),\ \ldots\}$$

Functions

Similarly, a ternary relation T is a (binary) function if for every pair of elements x_1 and x_2 there exists at most one (resp., exactly one for total function) y such that

$$(x_1, x_2, y) \in T.$$

Example: Addition on the set of real numbers $\mathbb{R}$ can be viewed as a ternary relation S (i.e., as a set of triples of real numbers) where

$$(x_1, x_2, y) \in S \qquad \text{iff} \qquad x_1 + x_2 = y$$

Functions

In predicate logic, functions can be expressed using predicates representing the corresponding relations — this is not very straightforward nor convenient.

Example: *"For each x and y it holds that $x + y \geq y + x$."*

$$\forall x \forall y \exists z \exists w (S(x, y, z) \ \wedge \ S(y, x, w) \ \wedge \ P(z, w))$$

- $S(x, y, z)$ — *"z is the sum of values x and y"*
- $P(x, y)$ — *"x greater than or equal to y"*

Remark: Moreover, we must assume that for every pair of elements x and y there exists exactly one element z such that $S(x, y, z)$.

Functions

In predicate logic, functions can be represented by **function symbols**.

Alphabet:

- ...
- function symbols: "f", "g", "h", ...
- ...

Every function symbol must have a specified **arity** corresponding to the arity of a function represented by this symbol (i.e., the number of arguments of this function).

Terms

Terms — expressions, consisting of variables, constant symbols, and function symbols; values of terms are elements of the universe

Example:

- Let us say that we have a predicate F where we assume that for every x there exists exactly one y such that

  $$F(x, y).$$

  Instead of binary predicate F, we can use unary function symbol f.

  Term

  $$f(x)$$

  represents this one particular element y, for which $F(x, y)$ holds.

  Instead of $\exists y(F(x, y) \wedge P(y))$, we can write $P(f(x))$.

Terms

Example:

- Let us say that we have a ternary predicate G where we assume that for every pair of elements x_1 and x_2 there exists exactly one y such that

  $$G(x_1, x_2, y).$$

  Instead of ternary predicate G, we can use binary function symbol g. Term

  $$g(x_1, x_2)$$

  represents this one particular element y, for which $G(x_1, x_2, y)$ holds.

Terms

Example: *"For each* x *and* y *it holds that* $x + y \geq y + x$*."*

$$\forall x \forall y P(f(x, y), f(y, x))$$

- f — binary function symbol where $f(x, y)$ represents the sum of values x and y
- P — binary predicate symbol where $P(x, y)$ represents relation *"x is greater than or equal to y"*

Terms

Variables, constant symbols and function symbols can be composed in terms in arbitrary way — it is only necessary to comply with the arity of the symbols (to apply each function symbol to a correct number of arguments).

Example:

- c — constant symbol
- f — unary function symbol
- g — binary function symbol
- h — binary function symbol

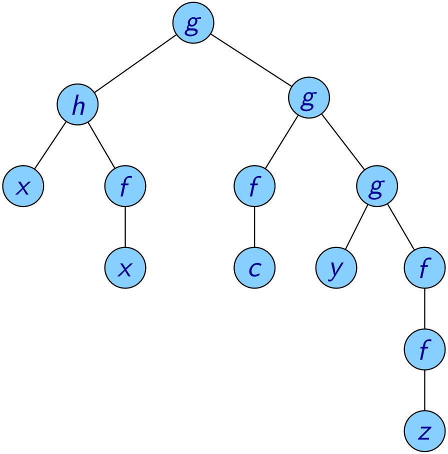
Examples of terms:

$$x \qquad f(y) \qquad g(c,x) \qquad g(h(x,x),f(c))$$

$$g(h(x,f(x)),g(f(c),g(y,f(f(z)))))$$

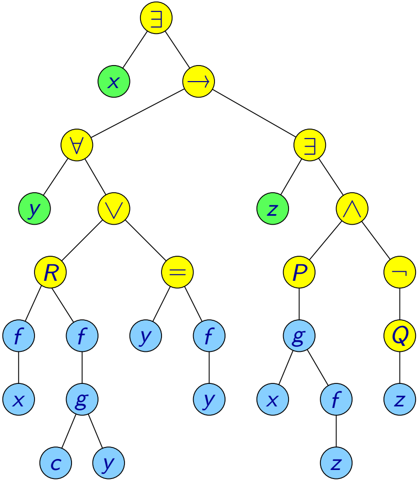
Terms

The syntactic tree of term $g(h(x, f(x)), g(f(c), g(y, f(f(z)))))$



Terms in Formulas

The syntactic tree of formula
$\exists x(\forall y(R(f(x), f(g(c, y))) \vee y = f(y)) \rightarrow \exists z(P(g(x, f(z))) \wedge \neg Q(z)))$



Terms in Formulas

Example:

- For each x, y, and z it holds that $(x+y)+z = x+(y+z)$:
$$\forall x \forall y \forall z (f(f(x,y),z) = f(x,f(y,z)))$$
- For each x it holds that $x+0=x$ and $0+x=x$:
$$\forall x (f(x,e) = x \ \wedge \ f(e,x) = x)$$
- For each x there exists y such that $x+y=0$:
$$\forall x \exists y (f(x,y) = e)$$

Constant and function symbols:

- f — binary function symbol representing *"addition"* (operation "$+$")
- e — constant symbol representing element "0"

Terms in Formulas

Example:

- For each x, y, and z it holds that $x \cdot (y + z) = x \cdot y + x \cdot z$:

$$\forall x \forall y \forall z (g(x, f(y, z)) = f(g(x, y), g(x, z)))$$

- For each x and y such that $x \leq y$ it holds that $x + z \leq y + z$:

$$\forall x \forall y (R(x, y) \ \rightarrow \ \forall z R(f(x, y), f(y, z)))$$

Constant and function symbols:

- f — binary function symbol representing *"addition"* (operation "$+$")
- g — binary function symbol representing *"multiplication"* (operation "$\cdot$")
- R — binary predicate symbol representing relation *"less than or equal to"* (relation "$\leq$")

Syntax of Formulas of Predicate Logic

Alphabet:

- **logical connectives** — "$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$"
- **quantifiers** — "$\forall$" and "$\exists$"
- **equality** — "$=$"
- **auxiliary symbols** — "$($", "$)$", and "$,$"
- **variables** — "x", "y", "z", …, "x_0", "x_1", "x_2", …

- **predicate symbols** — for example symbols "P", "Q", "R", etc. (for each symbols, there must be specified its arity)
- **function symbols** — for example symbols "f", "g", "h", etc. (for each symbol, there must be specified its arity)
- **constant symbols** — for example symbols "a", "b", "c", etc.

Remark: Constant symbols can be viewed as function symbols of arity 0.

Syntax of Formulas of Predicate Logic

Definition

Well-formed **terms** are defined as follows:

1. If x is a variable then x is a well-formed term.
2. If c is a constant symbol then c is a well-formed term.
3. If f is a function symbol of arity n and t_1, t_2, . . . , t_n are well-formed terms then
$$f(t_1, t_2, \ldots, t_n)$$
is a well-formed term.
4. There are no other well-formed terms than those constructed according to the previous rules.

Syntax of Formulas of Predicate Logic

Definition

Well-formed **atomic formulas** are defined as follows:

1. If P is a predicate symbol of arity n and t_1, t_2, . . . , t_n are well-formed terms then
$$P(t_1, t_2, \ldots, t_n)$$
is a well-formed atomic formula.
2. If t_1 and t_2 are well-formed terms then
$$t_1 = t_2$$
is a well-formed atomic formula.
3. There are no other well-formed atomic formulas than those constructed according to the previous rules.

Syntax of Formulas of Predicate Logic

Definition (a previously stated definition repeated)

Well-formed **formulas of predicate logic** are sequences of symbols constructed according to the following rules:

1. Well-formed atomic formulas are well-formed formulas.
2. If φ and ψ are well-formed formulas, then also $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ a $(\varphi \leftrightarrow \psi)$ are well-formed formulas.
3. If φ is a well-formed formula and x is a variable, then $\forall x \varphi$ and $\exists x \varphi$ are well-formed formulas.
4. There are no other well-formed formulas than those constructed according to the previous rules.

Semantics of Predicate Logic

Interpretation $\mathcal{A}$:

- universe A
- to every predicate symbol P of arity n, an n-ary relation $P^{\mathcal{A}}$ is assigned, where $P^{\mathcal{A}} \subseteq A \times A \times \cdots \times A$
- to every function symbol f of arity n, an n-ary function $f^{\mathcal{A}}$ is assigned, where $f^{\mathcal{A}} : A \times A \times \cdots \times A \to A$
- to every constant symbol c, an element of the universe $c^{\mathcal{A}}$ is assigned, i.e., $c^{\mathcal{A}} \in A$

Remark: In interpretations, only **total** functions, i.e., functions whose values are defined for all possible values of arguments, are assigned to function symbols.

Semantics of Predicate Logic

The **value of a term** in interpretation $\mathcal{A}$ and valuation v:

- Term x, where x is a variable — the value of this term is an element $a \in A$ such that $v(x) = a$.
- Term c, where c is a constant symbol — the value of this term is the element $c^{\mathcal{A}} \in A$.
- Term $f(t_1, t_2, \ldots, t_n)$, where $t_1, t_2, \ldots, t_n$ are terms — the value of this term is the element $b \in A$ such that

  $$b = f^{\mathcal{A}}(a_1, a_2, \ldots, a_n),$$

  where $a_1, a_2, \ldots, a_n$ are values of terms $t_1, t_2, \ldots, t_n$ in interpretation $\mathcal{A}$ and valuation v.

Semantics of Predicate Logic

Example: Interpretation $\mathcal{A}$ where the universe is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

- $a^{\mathcal{A}} = 0$
- $f^{\mathcal{A}}$ is the function *"successor"*, i.e., $f^{\mathcal{A}}(x) = x + 1$
- $g^{\mathcal{A}}$ is the function *"sum"*, i.e., $g^{\mathcal{A}}(x, y) = x + y$

Valuation v where $v(x) = 5$, $v(y) = 13$, $v(z) = 2$, ...

Values of terms in interpretation $\mathcal{A}$ and valuation v:

- Term x — value 5
- Term a — value 0
- Term $f(a)$ — value 1 $(0 + 1 = 1)$
- Term $f(f(a))$ — value 2 $(1 + 1 = 2)$
- Term $g(x, f(f(a)))$ — value 7 $(5 + 2 = 7)$
- Term $g(z, y)$ — value 15 $(2 + 13 = 15)$
- Term $f(g(z, y))$ — value 16 $(15 + 1 = 16)$

Semantics of Predicate Logic

The **truth of atomic formulas** in interpretation $\mathcal{A}$ and valuation v:

- $\mathcal{A}, v \models P(t_1, t_2, \ldots, t_n)$, where P is a predicate symbol of arity n and where $t_1, t_2, \ldots, t_n$ are terms, holds iff

  $$(a_1, a_2, \ldots, a_n) \in P^{\mathcal{A}},$$

  where $a_1, a_2, \ldots, a_n$ are values of terms $t_1, t_2, \ldots, t_n$ in interpretation $\mathcal{A}$ and valuation v.

- $\mathcal{A}, v \models t_1 = t_2$, where t_1 and t_2 are terms, holds iff

  $$a_1 = a_2,$$

  where a_1 and a_2 are values of terms t_1 and t_2 in interpretation $\mathcal{A}$ and valuation v.

Semantics of Predicate Logic

Example: Interpretation $\mathcal{A}$ where the universe is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

- $f^{\mathcal{A}}$ is the function *"successor"*, i.e., $f^{\mathcal{A}}(x) = x + 1$
- $g^{\mathcal{A}}$ is the function *"sum"*, i.e., $g^{\mathcal{A}}(x, y) = x + y$
- $P^{\mathcal{A}}$ is the set of all primes
- $Q^{\mathcal{A}}$ is the binary relation "$<$" i.e., $(x, y) \in Q^{\mathcal{A}}$ iff $x < y$

Valuation v where $v(x) = 5$, $v(y) = 13$, $v(z) = 2$, . . .

- $\mathcal{A}, v \models P(x)$ $\quad$ (5 is a prime)
- $\mathcal{A}, v \not\models Q(y, z)$ $\quad$ (it is not the case that $13 < 2$)
- $\mathcal{A}, v \models Q(f(f(z)), g(x, y))$ $\quad$ ($(2 + 1) + 1 < 5 + 13$)
- $\mathcal{A}, v \not\models P(f(g(z, x)))$ $\quad$ ($(2 + 5) + 1 = 8$ and 8 is not a prime)

Predicate Logic — Additional Comments

- The logic described here is **the first order** predicate logic — it is possible to quantify only over the elements of the universe (in the second order predicate logic, it is possible to quantify over relations).

Predicate Logic — Additional Comments

As commonly used in mathematics, it is ofter the case that formulas are not written according to the precise syntax of predicate logic but many kinds of conventions and abbreviations are used.

- For **binary** function and predicate symbols, it is common to use **infix** notation:

  For example, $f(x, y)$ and $R(x, y)$ can be written as

  $$x \ f \ y \qquad\qquad x \ R \ y$$

- To denote predicate, function and constant symbols, many different kinds of symbols are used:

  For example, $R(f(x, y), g(z))$ can be written as

  $x + \delta \leq |\varepsilon|$ or for example as $x \circ y \sqsupset G(z)$

Predicate Logic — Additional Comments

Examples of formulas representing propositions from set theory:

- x is an elements of set A:

  $$x \in A$$

  "$\in$" — binary predicate symbol representing the relation *"to be an element of"*
  "x", "A" — variables

  If we would do the following changes:

  - instead of symbol "$\in$", we would use binary predicate symbol E,
  - instead of variable A we would use variable y,

  then the formula would look as follows:

  $$E(x, y)$$

Predicate Logic — Additional Comments

- Two sets are equal if they contain the same elements:

$$A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B)$$

If we use predicate E instead of "$\in$", and y and z instead of A and B, the formula would look as follows:

$$y = z \leftrightarrow \forall x(E(x,y) \leftrightarrow E(x,z))$$

- The definition of relation *"to be a subset"* (denoted by symbol "$\subseteq$"):

$$A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$$

If we use binary predicate symbol S instead of "$\subseteq$":

$$S(y,z) \leftrightarrow \forall x(E(x,y) \rightarrow E(x,z))$$

Predicate Logic — Additional Comments

- The definition of operation *"union"* (denoted by symbol "$\cup$"):

  $$\forall x(x \in A \cup B \;\leftrightarrow\; (x \in A \;\vee\; x \in B))$$

  If we use binary function symbol f instead of "$\cup$":

  $$\forall x(E(x, f(y,z)) \;\leftrightarrow\; (E(x,y) \;\vee\; E(x,z)))$$

Predicate Logic — Additional Comments

- Sometimes

$$\exists x (x \in A \ \wedge \ \ldots \ )$$

is written in an abbreviated way as

$$(\exists x \in A)(\ \ldots \ )$$

I.e., instead of

"there exists x such that $x \in A$ and . . ."

we can say

"there exists $x \in A$ such that . . ."

- Similarly, $\exists x (x \geq 1 \ \wedge \ \ldots)$ can be written in an abbreviated way as

$$(\exists x \geq 1)(\ \ldots \ )$$

Predicate Logic — Additional Comments

- Sometimes

  $$\forall x(x \in A \ \rightarrow \ \ldots\ )$$

  is written in an abbreviated way as

  $$(\forall x \in A)(\ \ldots\ )$$

  I.e., instead of

  "for each x, for which $x \in A$ holds, we have . . . "

  we can say

  "for each $x \in A$ we have . . . "

- Similarly, $\forall x(x \geq 1 \ \rightarrow \ \ldots)$ can be written in an abbreviated way as

  $$(\forall x \geq 1)(\ \ldots\ )$$

Formal Languages

Alphabet and Word

Definition

Alphabet is a nonempty finite set of **symbols**.

Remark: An alphabet is often denoted by the symbol Σ (upper case sigma) of the Greek alphabet.

Definition

A **word** over a given alphabet is a finite sequence of symbols from this alphabet.

Example 1:

$$\Sigma = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}, \mathtt{F}, \mathtt{G}, \mathtt{H}, \mathtt{I}, \mathtt{J}, \mathtt{K}, \mathtt{L}, \mathtt{M}, \mathtt{N}, \mathtt{O}, \mathtt{P}, \mathtt{Q}, \mathtt{R}, \mathtt{S}, \mathtt{T}, \mathtt{U}, \mathtt{V}, \mathtt{W}, \mathtt{X}, \mathtt{Y}, \mathtt{Z}\}$$

Words over alphabet Σ: $\mathtt{HELLO}$ $\mathtt{ABRACADABRA}$ $\mathtt{ERROR}$

Alphabet and Word

Example 2:

$\Sigma_2 = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}, \mathtt{F}, \mathtt{G}, \mathtt{H}, \mathtt{I}, \mathtt{J}, \mathtt{K}, \mathtt{L}, \mathtt{M}, \mathtt{N}, \mathtt{O}, \mathtt{P}, \mathtt{Q}, \mathtt{R}, \mathtt{S}, \mathtt{T}, \mathtt{U}, \mathtt{V}, \mathtt{W}, \mathtt{X}, \mathtt{Y}, \mathtt{Z}, \sqcup\}$

A word over alphabet Σ_2: $\mathtt{HELLO}\sqcup\mathtt{WORLD}$

Example 3:

$\Sigma_3 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Words over alphabet Σ_3: 0, 31415926536, 65536

Example 4:

Words over alphabet $\Sigma_4 = \{0, 1\}$: 011010001, 111, 1010101010101010

Example 5:

Words over alphabet $\Sigma_5 = \{a, b\}$: *aababb*, *abbabbba*, *aaab*

Alphabet and Word

Example 6:

Alphabet Σ_6 is the set of all ASCII characters.

Example of a word:

```
class HelloWorld {
    public static void main(String[] args) {
        System.out.println("Hello, world!");
    }
}
```

class␣HelloWorld␣{ ↩ ␣␣␣␣public␣static␣void␣main(Str···

Theory of Formal Languages – Motivation

Language — a set of (some) words of symbols from a given alphabet

Examples of problem types, where theory of formal languages is useful:

- Construction of compilers:
  - Lexical analysis
  - Syntactic analysis
- Searching in text:
  - Searching for a given text pattern
  - Seaching for a part of text specified by a regular expression

Representation of Formal Languages

To describe a language, there are several possibilities:

- We can enumerate all words of the language (however, this is possible only for small finite languages).

  Example: $L = \{aab, babba, aaaaaa\}$

- We can specify a property of the words of the language:

  Example: The language over alphabet $\{0, 1\}$ containing all words with even number of occurrences of symbol 1.

Representation of Formal Languages

In particular, the following two approaches are used in the theory of formal languages:

- To describe an (idealized) machine, device, algorithm, that recognizes words of the given language – approaches based on **automata**.
- To describe some mechanism that allows to generate all words of the given language – approaches based on **grammars** or **regular expressions**.

Some Basic Concepts

The **length of a word** is the number of symbols of the word.

For example, the length of word $abaab$ is 5.

The length of a word w is denoted $|w|$.

For example, if $w = abaab$ then $|w| = 5$.

We denote the number of occurrences of a symbol a in a word w by $|w|_a$.

For word $w = ababb$ we have $|w|_a = 2$ and $|w|_b = 3$.

An **empty word** is a word of length 0, i.e., the word containing no symbols.

The empty word is denoted by the letter ε (epsilon) of the Greek alphabet.

(Remark: In literature, sometimes λ (lambda) is used to denote the empty word instead of ε .)

$$|\varepsilon| = 0$$

Concatenation of Words

One of operations we can do on words is the operation of **concatenation**:

For example, the concatenation of words $\mathtt{OST}$ and $\mathtt{RAVA}$ is the word $\mathtt{OSTRAVA}$.

The operation of concatenation is denoted by symbol $\cdot$ (similarly to multiplication). It is possible to omit this symbol.

$$\mathtt{OST} \cdot \mathtt{RAVA} = \mathtt{OSTRAVA}$$

Concatenation is **associative**, i.e., for every three words u, v, and w we have

$$(u \cdot v) \cdot w = u \cdot (v \cdot w)$$

which means that we can omit parenthesis when we write multiple concatenations. For example, we can write $w_1 \cdot w_2 \cdot w_3 \cdot w_4 \cdot w_5$ instead of $(w_1 \cdot (w_2 \cdot w_3)) \cdot (w_4 \cdot w_5)$.

Concatenation of Words

Concatenation is not **commutative**, i.e., the following equality does not hold in general

$$u \cdot v = v \cdot u$$

Example:

$$\mathtt{OST} \cdot \mathtt{RAVA} \neq \mathtt{RAVA} \cdot \mathtt{OST}$$

It is obvious that the following holds for any words v and w:

$$|v \cdot w| = |v| + |w|$$

For every word w we also have:

$$\varepsilon \cdot w = w \cdot \varepsilon = w$$

Prefixes, Suffixes, and Subwords

Definition

A word x is a **prefix** of a word y, if there exists a word v such that $y = xv$.

A word x is a **suffix** of a word y, if there exists a word u such that $y = ux$.

A word x is a **subword** of a word y, if there exist words u and v such that $y = uxv$.

Example:

- Prefixes of the word `abaab` are ε, `a`, `ab`, `aba`, `abaa`, `abaab`.
- Suffixes of the word `abaab` are ε, `b`, `ab`, `aab`, `baab`, `abaab`.
- Subwords of the word `abaab` are ε, `a`, `b`, `ab`, `ba`, `aa`, `aba`, `baa`, `aab`, `abaa`, `baab`, `abaab`.

Language

The set of all words over alphabet Σ is denoted Σ^*.

Definition

A **(formal) language** L over an alphabet Σ is a subset of Σ^*, i.e., $L \subseteq \Sigma^*$.

Example 1: The set $\{00, 01001, 1101\}$ is a language over alphabet $\{0, 1\}$.

Example 2: The set of all syntactically correct programs in the C programming language is a language over the alphabet consisting of all ASCII characters.

Example 3: The set of all texts containing the sequence `hello` is a language over alphabet consisting of all ASCII characters.

Set Operations on Languages

Since languages are sets, we can apply any set operations to them:

Union – $L_1 \cup L_2$ is the language consisting of the words belonging to language L_1 or to language L_2 (or to both of them).

Intersection – $L_1 \cap L_2$ is the language consisting of the words belonging to language L_1 and also to language L_2.

Complement – $\overline{L_1}$ is the language containing those words from Σ^* that do not belong to L_1.

Difference – $L_1 - L_2$ is the language containing those words of L_1 that do not belong to L_2.

Remark: It is assumed the languages involved in these operations use the same alphabet Σ.

Set Operations on Languages

Formally:

Union: $L_1 \cup L_2 = \{w \in \Sigma^* \mid w \in L_1 \vee w \in L_2\}$

Intersection: $L_1 \cap L_2 = \{w \in \Sigma^* \mid w \in L_1 \wedge w \in L_2\}$

Complement: $\overline{L_1} = \{w \in \Sigma^* \mid w \notin L_1\}$

Difference: $L_1 - L_2 = \{w \in \Sigma^* \mid w \in L_1 \wedge w \notin L_2\}$

Remark: We assume that $L_1, L_2 \subseteq \Sigma^*$ for some given alphabet Σ.

Set Operations on Languages

Example:

Consider languages over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

- L_1 — the set of all words containing subword baa
- L_2 — the set of all words with an even number of occurrences of symbol b

Then

- $L_1 \cup L_2$ — the set of all words containing subword baa or an even number of occurrences of b
- $L_1 \cap L_2$ — the set of all words containing subword baa and an even number of occurrences of b
- $\overline{L_1}$ — the set of all words that do not contain subword baa
- $L_1 - L_2$ — the set of all words that contain subword baa but do not contain an even number of occurrences of b

Concatenation of Languages

Definition

Concatenation of languages L_1 and L_2, where $L_1, L_2 \subseteq \Sigma^*$, is the language $L \subseteq \Sigma^*$ such that for each $w \in \Sigma^*$ it holds that

$$w \in L \;\leftrightarrow\; (\exists u \in L_1)(\exists v \in L_2)(w = u \cdot v)$$

The concatenation of languages L_1 and L_2 is denoted $L_1 \cdot L_2$.

Example:

$$\begin{array}{rcl} L_1 & = & \{abb, ba\} \\ L_2 & = & \{a, ab, bbb\} \end{array}$$

The language $L_1 \cdot L_2$ contains the following words:

$abba \qquad abbab \qquad abbbbb \qquad baa \qquad baab \qquad babbb$

Iteration of a Language

Definition

The **iteration (Kleene star) of language** L, denoted L^*, is the language consisting of words created by concatenation of some arbitrary number of words from language L.
I.e. $w \in L^*$ iff

$$\exists n \in \mathbb{N} : \exists w_1, w_2, \ldots, w_n \in L : w = w_1 w_2 \cdots w_n$$

Example: $L = \{aa, b\}$

$$L^* = \{\varepsilon, aa, b, aaaa, aab, baa, bb, aaaaaa, aaaab, aabaa, aabb, \ldots\}$$

Remark: The number of concatenated words can be 0, which means that $\varepsilon \in L^*$ always holds (it does not matter if $\varepsilon \in L$ or not).

Iteration of a Language – Alternative Definition

At first, for a language L and a number $k \in \mathbb{N}$ we define the language L^k:

$$L^0 = \{\varepsilon\}, \qquad L^k = L^{k-1} \cdot L \quad \text{for } k \geq 1$$

This means

$$\begin{array}{rcl}
L^0 & = & \{\varepsilon\} \\
L^1 & = & L \\
L^2 & = & L \cdot L \\
L^3 & = & L \cdot L \cdot L \\
L^4 & = & L \cdot L \cdot L \cdot L \\
L^5 & = & L \cdot L \cdot L \cdot L \cdot L \\
 & & \cdots
\end{array}$$

Example: For $L = \{aa, b\}$, the language L^3 contains the following words:

$aaaaaa \quad aaaab \quad aabaa \quad aabb \quad baaaa \quad baab \quad bbaa \quad bbb$

Iteration of a Language – Alternative Definition

Alternative definition

The **iteration (Kleene star) of language** L is the language

$$L^* = \bigcup_{k \geq 0} L^k$$

Remark:

$$\bigcup_{k \geq 0} L^k = L^0 \cup L^1 \cup L^2 \cup L^3 \cup \cdots$$

Iteration of a Language

Remark: Sometimes, notation L^+ is used as an abbreviation for $L \cdot L^*$, i.e.,

$$L^+ = \bigcup_{k \geq 1} L^k$$

Reverse

The **reverse** of a word w is the word w written from backwards (in the opposite order).

The reverse of a word w is denoted w^R.

Example: $w = \mathtt{HELLO}$ $w^R = \mathtt{OLLEH}$

Formally, for $w = a_1 a_2 \cdots a_n$ (where $a_i \in \Sigma$) is $w^R = a_n a_{n-1} \cdots a_1$.

Reverse

The **reverse** of a language L is the language consisting of reverses of all words of L.

Reverse of a language L is denoted L^R.

$$L^R = \{w^R \mid w \in L\}$$

Example: $L = \{ab, baaba, aaab\}$
$L^R = \{ba, abaab, baaa\}$

Order on Words

Let us assume some (linear) order $<$ on the symbols of alphabet Σ, i.e., if $\Sigma = \{a_1, a_2, \ldots, a_n\}$ then

$$a_1 < a_2 < \ldots < a_n \,.$$

Example: $\Sigma = \{a, b, c\}$ with $a < b < c$.

The following (linear) order $<_L$ can be defined on Σ^*:
$x <_L y$ iff:

- $|x| < |y|$, or
- $|x| = |y|$ there exist words $u, v, w \in \Sigma^*$ and symbols $a, b \in \Sigma$ such that

$$x = uav \qquad y = ubw \qquad a < b$$

Informally, we can say that in order $<_L$ we order words according to their length, and in case of the same length we order them lexicographically.

Order on Words

All words over alphabet Σ can be ordered by $<_L$ into a sequence

$$w_0, w_1, w_2, \ldots$$

where every word $w \in \Sigma^*$ occurs exactly once, and where for each $i, j \in \mathbb{N}$ it holds that $w_i <_L w_j$ iff $i < j$.

Example: For alphabet $\Sigma = \{a, b, c\}$ (where $a < b < c$) , the initial part of the sequence looks as follows:

$$\varepsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, aac, aba, abb, abc, \ldots$$

For example, when we talk about the first ten words of a language $L \subseteq \Sigma^*$, we mean ten words that belong to language L and that are smallest of all words of L according to order $<_L$.

Regular Expressions

Regular Expressions

Regular expressions describing languages over an alphabet Σ:

- $\emptyset$, ε, a (where $a \in \Sigma$) are regular expressions:

  $\emptyset$ ... denotes the empty language
  ε ... denotes the language $\{\varepsilon\}$
  a ... denotes the language $\{a\}$

- If α, β are regular expressions then also $(\alpha + \beta)$, $(\alpha \cdot \beta)$, (α^*) are regular expressions:

  $(\alpha + \beta)$ ... denotes the union of languages denoted α and β
  $(\alpha \cdot \beta)$ ... denotes the concatenation of languages denoted α and β
  (α^*) ... denotes the iteration of a language denoted α

- There are no other regular expressions except those defined in the two points mentioned above.

Regular Expressions

Example: alphabet $\Sigma = \{0, 1\}$

- According to the definition, $\mathtt{0}$ and $\mathtt{1}$ are regular expressions.

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- Since 0 is a regular expression, (0^*) is also a regular expression.
- Since $(0 + 1)$ and (0^*) are regular expressions, $((0 + 1) \cdot (0^*))$ is also a regular expression.

Regular Expressions

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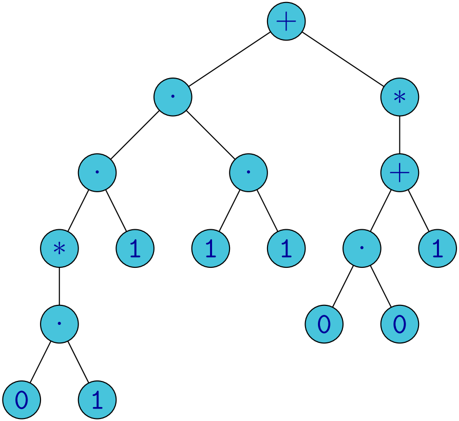
- Since 0 is a regular expression, (0^*) is also a regular expression.
- Since $(0 + 1)$ and (0^*) are regular expressions, $((0 + 1) \cdot (0^*))$ is also a regular expression.

Remark: If α is a regular expression, by $[\alpha]$ we denote the language defined by the regular expression α.

$$[((0 + 1) \cdot (0^*))] = \{0, 1, 00, 10, 000, 100, 0000, 1000, 00000, \ldots\}$$

Regular Expressions

The structure of a regular expression can be represented by an abstract syntax tree:



$$(((((0 \cdot 1)^*) \cdot 1) \cdot (1 \cdot 1)) + (((0 \cdot 0) + 1)^*))$$

Regular Expressions

The formal definition of semantics of regular expressions:

- $[\emptyset] = \emptyset$
- $[\varepsilon] = \{\varepsilon\}$
- $[a] = \{a\}$
- $[\alpha^*] = [\alpha]^*$
- $[\alpha \cdot \beta] = [\alpha] \cdot [\beta]$
- $[\alpha + \beta] = [\alpha] \cup [\beta]$

Regular Expressions

To make regular expressions more lucid and succinct, we use the following conventions:

- The outward pair of parentheses can be omitted.
- We can omit parentheses that are superflous due to associativity of operations of union ($+$) and concatenation ($\cdot$).
- We can omit parentheses that are superflous due to the defined priority of operators (iteration (*) has the highest priority, concatenation ($\cdot$) has lower priority, and union ($+$) has the lowest priority).
- A dot denoting concatenation can be omitted.

Example: Instead of

$$(((((0 \cdot 1)^*) \cdot 1) \cdot (1 \cdot 1)) + (((0 \cdot 0) + 1)^*))$$

we usually write

$$(01)^*111 + (00 + 1)^*$$

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

$\mathtt{0}$ ... the language containing the only word $\mathtt{0}$

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

$\mathtt{0}$ ... the language containing the only word $\mathtt{0}$

$\mathtt{01}$ ... the language containing the only word $\mathtt{01}$

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 ... the language containing the only word 0

01 ... the language containing the only word 01

$0 + 1$ ... the language containing two words 0 and 1

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 . . . the language containing the only word 0

01 . . . the language containing the only word 01

$0 + 1$ . . . the language containing two words 0 and 1

0^* . . . the language containing words ε, 0, 00, 000, . . .

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 ... the language containing the only word 0

01 ... the language containing the only word 01

$0 + 1$ ... the language containing two words 0 and 1

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$(01)^*$ ... the language containing words ε, 01, 0101, 010101, ...

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

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$(0 + 1)^*$ ... the language containing all words over the alphabet $\{0, 1\}$

Regular Expressions

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$(0 + 1)^*00$ ... the language containing all words ending with 00

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$(0 + 1)^*00$ ... the language containing all words ending with 00

$(01)^*111(01)^*$ ... the language containing all words that contain a subword 111 preceded and followed by an arbitrary number of copies of the word 01

Regular Expressions

$(0+1)^*00+(01)^*111(01)^*$ . . . the language containing all words that either end with 00 or contain a subwords 111 preceded and followed with some arbitrary number of words 01

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Regular Expressions

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$(0+1)^*1(0+1)^*$ . . . the language of all words that contain at least one occurrence of symbol 1

$0^*(10^*10^*)^*$ . . . the language containg all words with an even number of occurrences of symbol 1

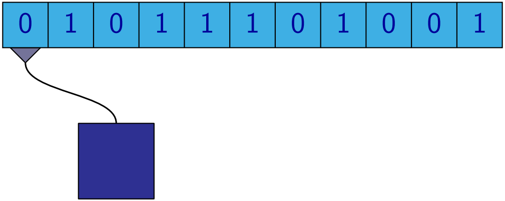
Finite Automata

Recognition of a Language

Example: Consider words over alphabet $\{0, 1\}$.

We would like to recognize a language L consisting of words with even number of symbols 1.

We want to design a device that reads a word and then tells us if the word belongs to the language L or not.



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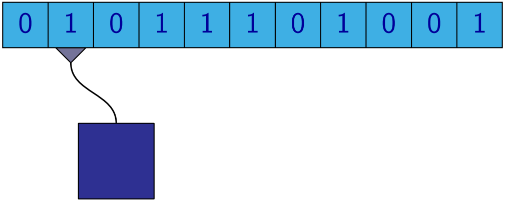

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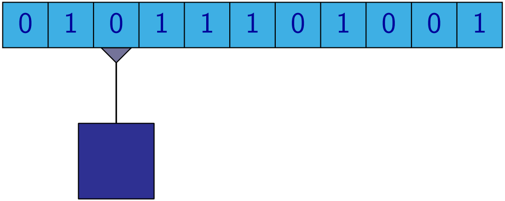

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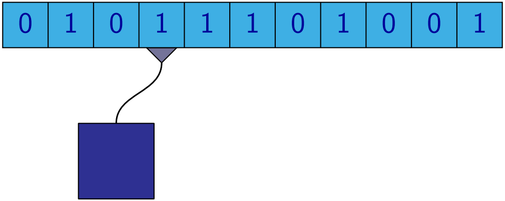

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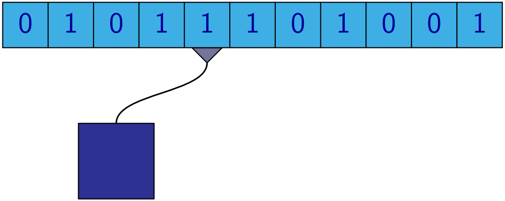

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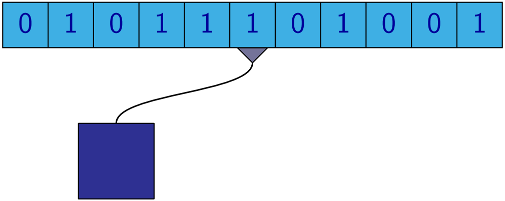

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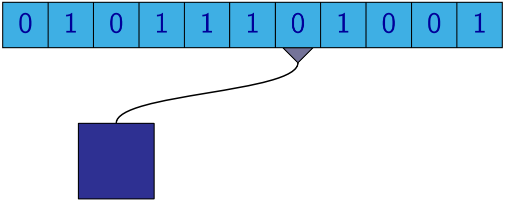

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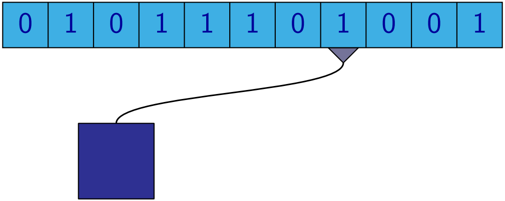

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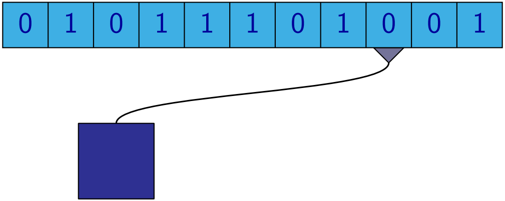

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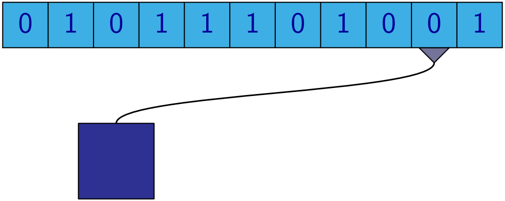

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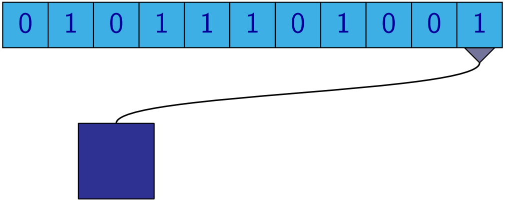

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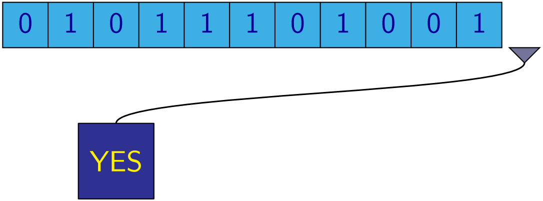
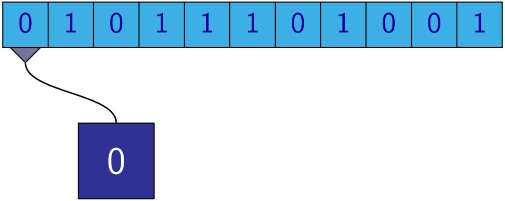


Recognition of a Language

The first idea: To count the number of occurrences of symbol 1.



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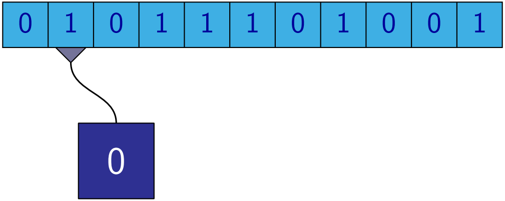

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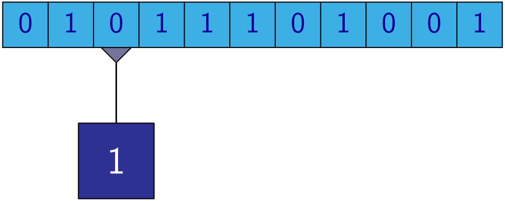

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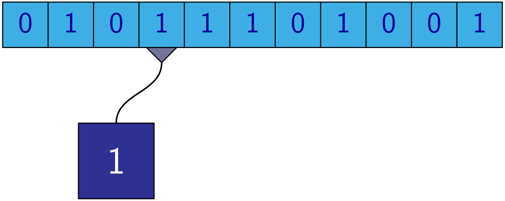

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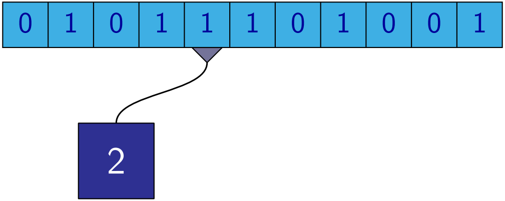

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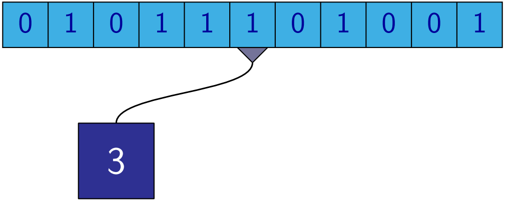

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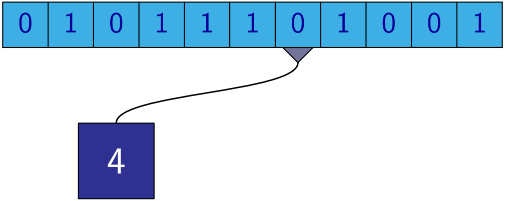

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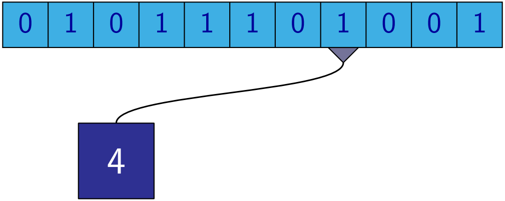

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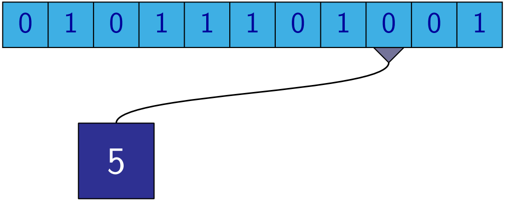

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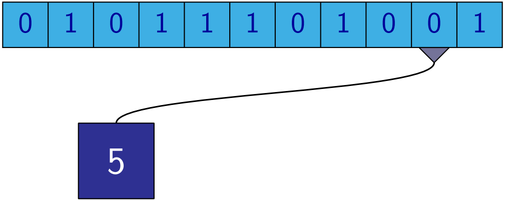

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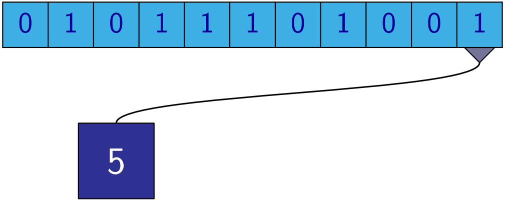

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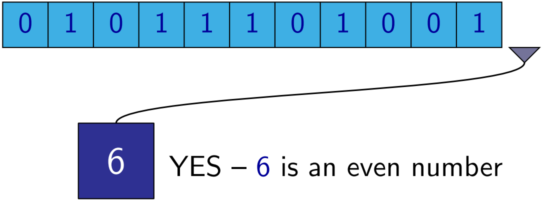
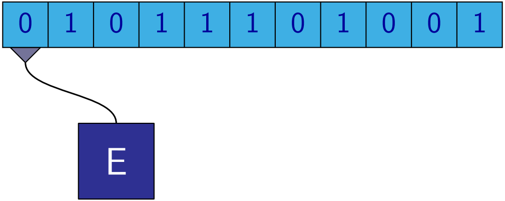


0	1	0	1	1	1	0	1	0	0	1

YES – 6 is an even number

Recognition of a Language

The second idea: In fact, we just need to remember if the number of symbols 1 read so far is even or odd (i.e., it is sufficient to remember only the last bit of the number).



0 1 0 1 1 1 0 1 0 0 1

E

Recognition of a Language

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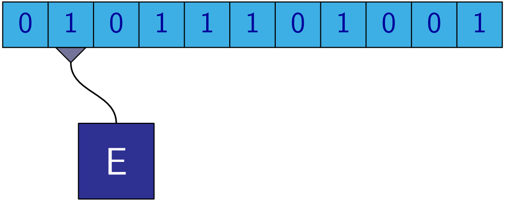

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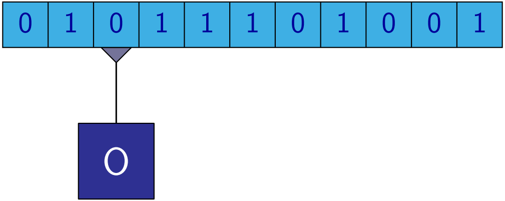

Recognition of a Language

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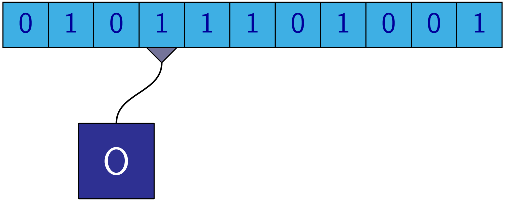

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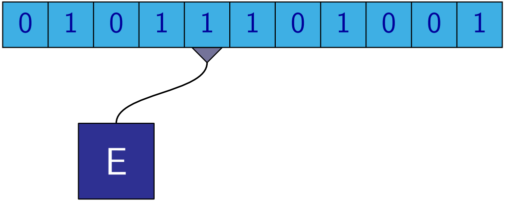

Recognition of a Language

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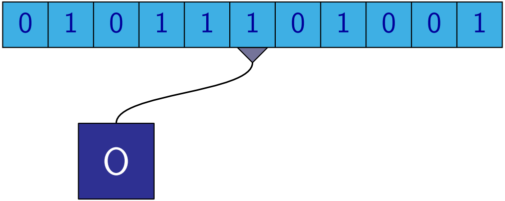

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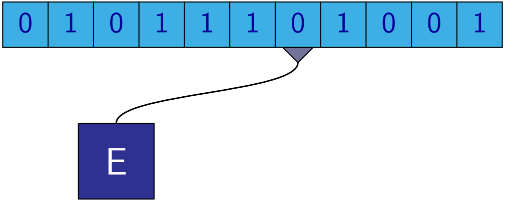

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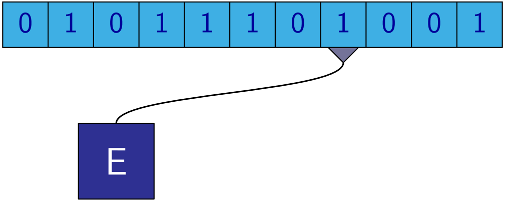

Recognition of a Language

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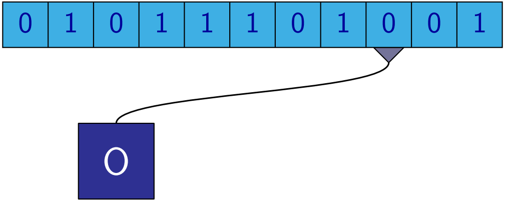
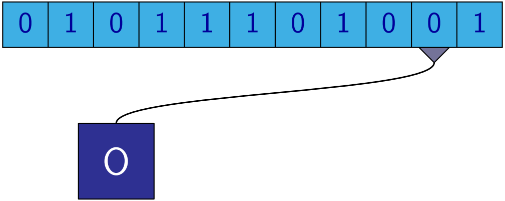


0 1 0 1 1 1 0 1 0 0 1

O

Recognition of a Language

The second idea: In fact, we just need to remember if the number of symbols 1 read so far is even or odd (i.e., it is sufficient to remember only the last bit of the number).



Recognition of a Language

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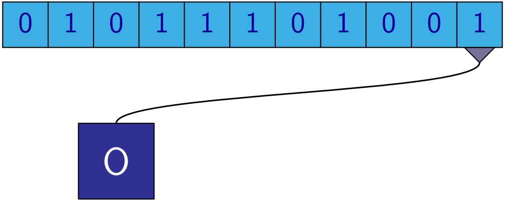

Recognition of a Language

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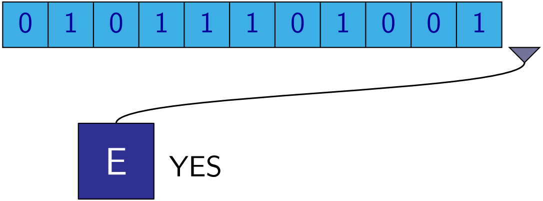

Recognition of a Language

The behaviour of the device can be described by the following graph:

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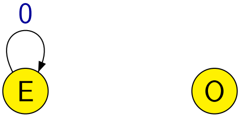

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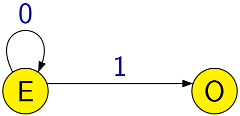

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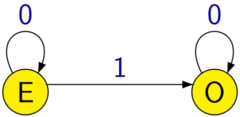

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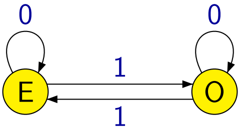

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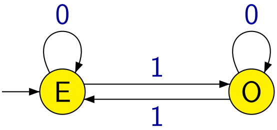

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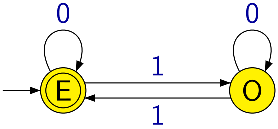

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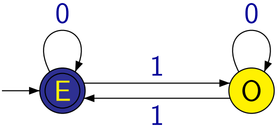
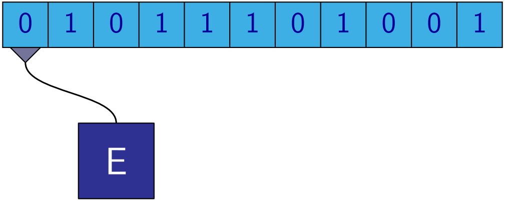
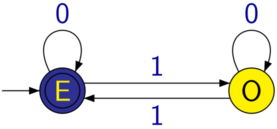


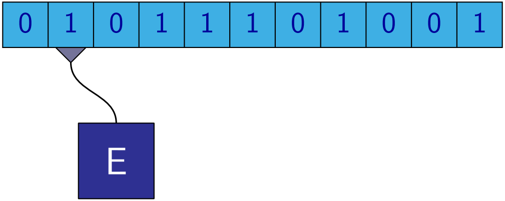


0	1	0	1	1	1	0	1	0	0	1

Recognition of a Language

The behaviour of the device can be described by the following graph:





0	1	0	1	1	1	0	1	0	0	1

Recognition of a Language

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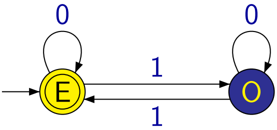

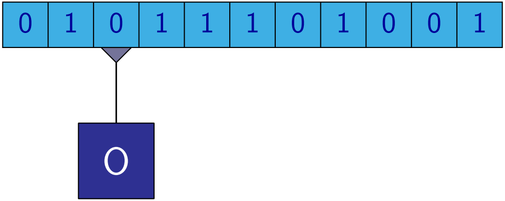

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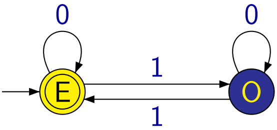

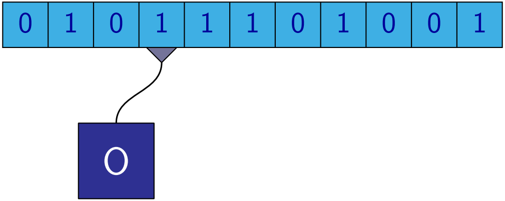

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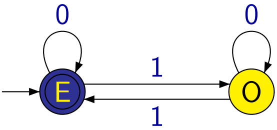

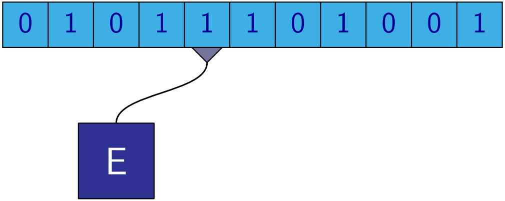

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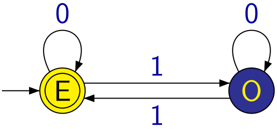

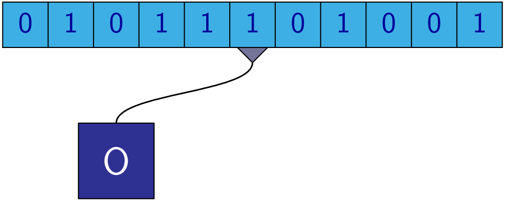

0	1	0	1	1	1	0	1	0	0	1

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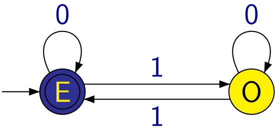

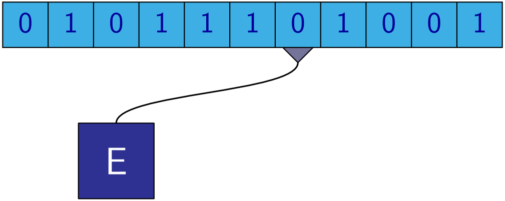

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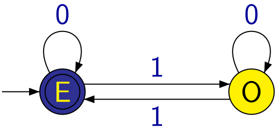

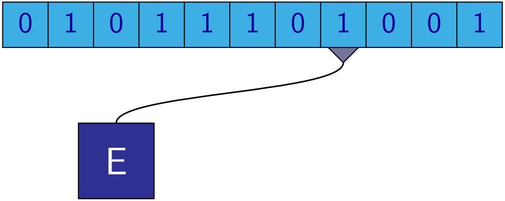

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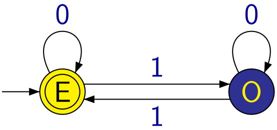

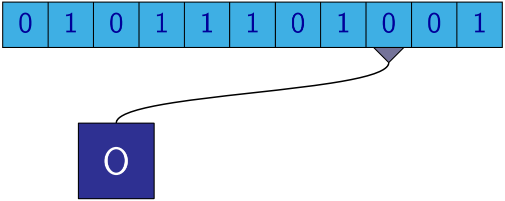

0	1	0	1	1	1	0	1	0	0	1

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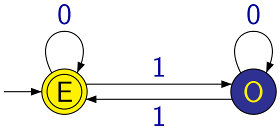

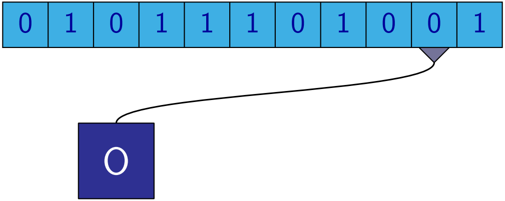

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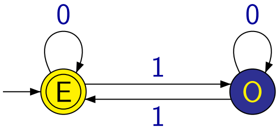

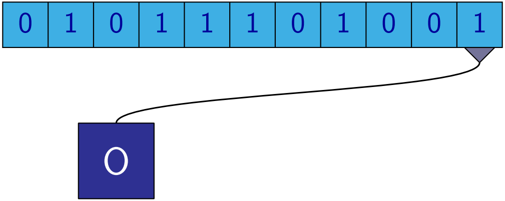

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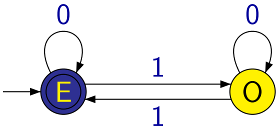
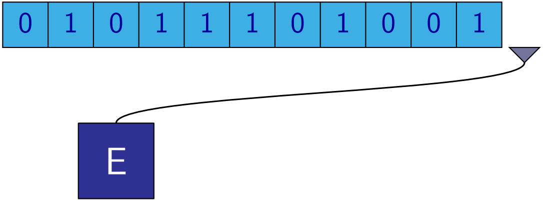
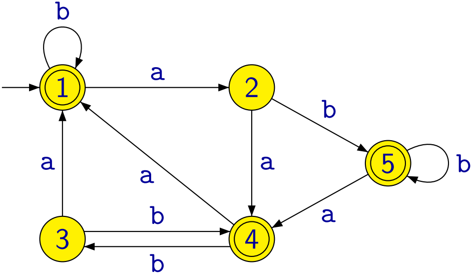




Deterministic Finite Automaton



A **deterministic finite automaton** consists of **states** and **transitions**. One of the states is denoted as an **initial state** and some of states are denoted as **accepting**.

Deterministic Finite Automaton

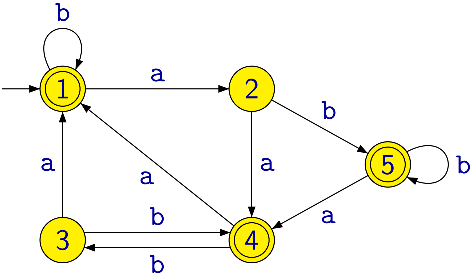
Formally, a **deterministic finite automaton** (**DFA**) is defined as a tuple

$$(Q, \Sigma, \delta, q_0, F)$$

where:

- Q is a nonempty finite set of **states**
- Σ is an **alphabet** (a nonempty finite set of symbols)
- $\delta : Q \times \Sigma \rightarrow Q$ is a **transition function**
- $q_0 \in Q$ is an **initial state**
- $F \subseteq Q$ is a set of **accepting states**

Deterministic Finite Automaton



- $Q = \{1, 2, 3, 4, 5\}$
- $\Sigma = \{\mathtt{a}, \mathtt{b}\}$
- $q_0 = 1$
- $F = \{1, 4, 5\}$

$$
\begin{array}{ll}
\delta(1, \mathtt{a}) = 2 & \delta(1, \mathtt{b}) = 1 \\
\delta(2, \mathtt{a}) = 4 & \delta(2, \mathtt{b}) = 5 \\
\delta(3, \mathtt{a}) = 1 & \delta(3, \mathtt{b}) = 4 \\
\delta(4, \mathtt{a}) = 1 & \delta(4, \mathtt{b}) = 3 \\
\delta(5, \mathtt{a}) = 4 & \delta(5, \mathtt{b}) = 5
\end{array}
$$

Deterministic Finite Automaton

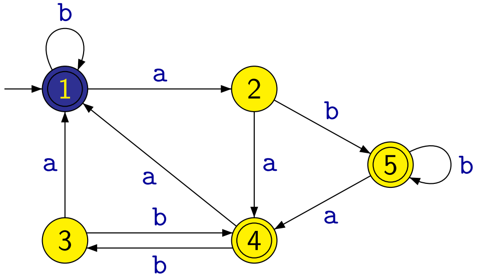
Instead of

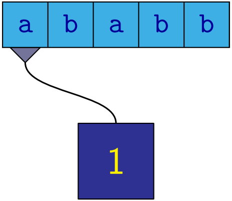
$$\begin{array}{ll}
\delta(1,\mathtt{a}) = 2 & \delta(1,\mathtt{b}) = 1 \\
\delta(2,\mathtt{a}) = 4 & \delta(2,\mathtt{b}) = 5 \\
\delta(3,\mathtt{a}) = 1 & \delta(3,\mathtt{b}) = 4 \\
\delta(4,\mathtt{a}) = 1 & \delta(4,\mathtt{b}) = 3 \\
\delta(5,\mathtt{a}) = 4 & \delta(5,\mathtt{b}) = 5
\end{array}$$

we rather use a more succinct representation as a table or a depicted graph:

δ	$\mathtt{a}$	$\mathtt{b}$
$\leftrightarrow 1$	2	1
2	4	5
3	1	4
$\leftarrow 4$	1	3
$\leftarrow 5$	4	5

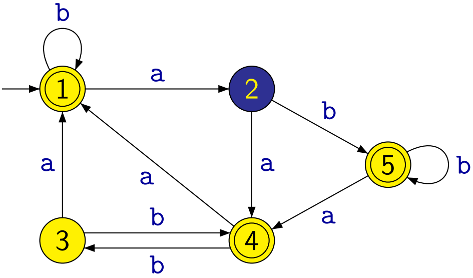
Deterministic Finite Automaton

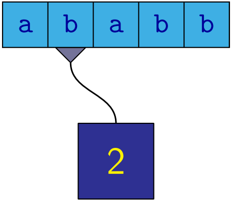




1

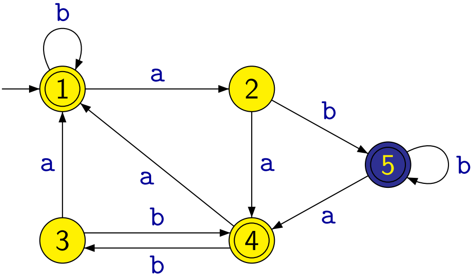
Deterministic Finite Automaton

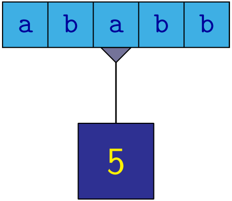




$1 \xrightarrow{\texttt{a}} 2$

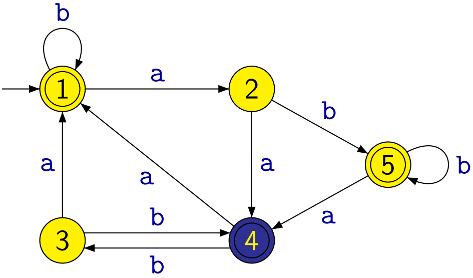
Deterministic Finite Automaton

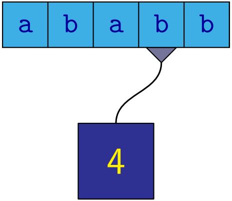




$$1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{b}} 5$$

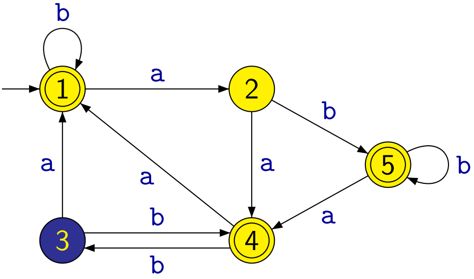
Deterministic Finite Automaton

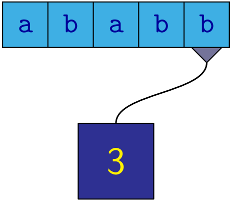




$$1 \xrightarrow{\mathtt{a}} 2 \xrightarrow{\mathtt{b}} 5 \xrightarrow{\mathtt{a}} 4$$

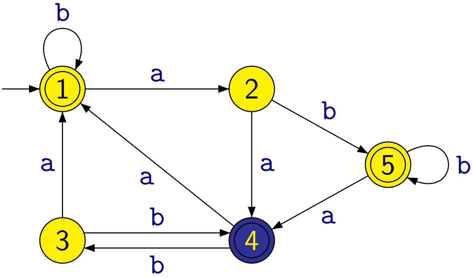
Deterministic Finite Automaton

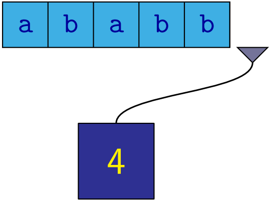




$$1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{b}} 5 \xrightarrow{\texttt{a}} 4 \xrightarrow{\texttt{b}} 3$$

Deterministic Finite Automaton





$$1 \xrightarrow{a} 2 \xrightarrow{b} 5 \xrightarrow{a} 4 \xrightarrow{b} 3 \xrightarrow{b} 4$$

Deterministic Finite Automaton

Definition

Let us have a DFA $A = (Q, \Sigma, \delta, q_0, F)$.

By $q \xrightarrow{w} q'$, where $q, q' \in Q$ and $w \in \Sigma^*$, we denote the fact that the automaton, starting in state q goes to state q' by reading word w.

Remark: $\longrightarrow \subseteq Q \times \Sigma^* \times Q$ is a ternary relation.

Instead of $(q, w, q') \in \longrightarrow$ we write $q \xrightarrow{w} q'$.

It holds for a DFA that for each state q and each word w there is exactly one state q' such that $q \xrightarrow{w} q'$.

Deterministic Finite Automaton

Relation $\longrightarrow$ can be formally defined by the following inductive definition:

- $q \xrightarrow{\varepsilon} q$ for each $q \in Q$
- For $a \in \Sigma$ and $w \in \Sigma^*$:
  $q \xrightarrow{aw} q'$ iff there is $q'' \in Q$ such that $\delta(q, a) = q''$ and $q'' \xrightarrow{w} q'$.

Deterministic Finite Automaton

A word $w \in \Sigma^*$ is **accepted** by a deterministic finite automaton $A = (Q, \Sigma, \delta, q_0, F)$ iff there exists a state $q \in F$ such that $q_0 \xrightarrow{w} q$.

Definition

A **language** accepted by a given deterministic finite automaton $A = (Q, \Sigma, \delta, q_0, F)$, denoted $L(A)$, is the set of all words accepted by the automaton, i.e.,

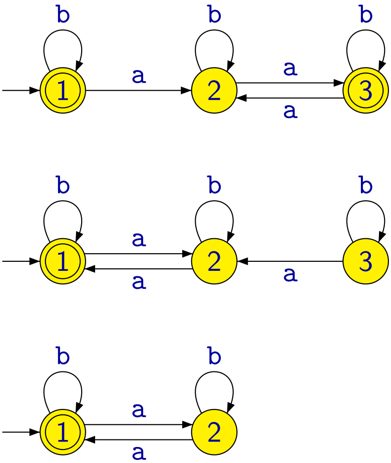
$$L(A) = \{w \in \Sigma^* \mid \exists q \in F : q_0 \xrightarrow{w} q\}$$

Regular languages

Definition

A language L is **regular** iff there exists some deterministic finite automaton accepting L, i.e., DFA A such that $L(A) = L$.

Equivalence of Automata



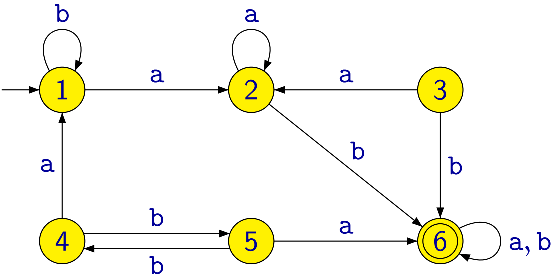
All 3 automata accept the language of all words with an even number of *a*'s.

Equivalence of Automata

Definition

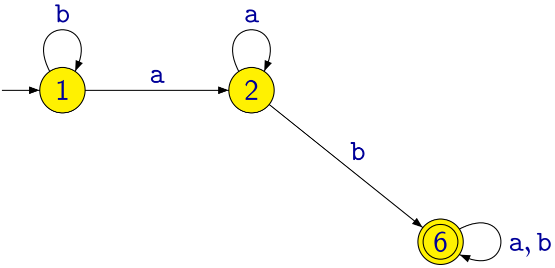
We say automata A_1, A_2 are **equivalent** if $L(A_1) = L(A_2)$.

Unreachable States of an Automaton



- The automaton accepts the language
  $L = \{w \in \{a, b\}^* \mid w \text{ contains subword } ab\}$
- There is no input sequence such that after reading it, the automaton gets to states 3, 4, or 5.

Unreachable States of an Automaton



- The automaton accepts the language
  $L = \{w \in \{a, b\}^* \mid w \text{ contains subword } ab\}$
- There is no input sequence such that after reading it, the automaton gets to states 3, 4, or 5.
- If we remove these states, the automaton still accepts the same language L.

Unreachable States of an Automaton

Definition

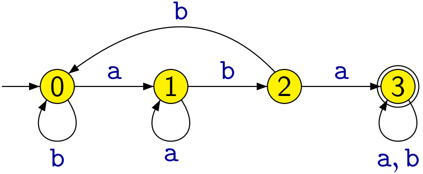
A state q of a finite automaton $A = (Q, \Sigma, \delta, q_0, F)$ is **reacheable** if there exists a word w such that $q_0 \xrightarrow{w} q$.

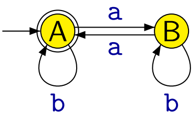
Otherwise the state is **unreachable**.

- There is no path in a graph of an automaton going from the initial state to some unreachable state.
- Unreachable states can be removed from an automaton (together with all transitions going to them and from them). The language accepted by the automaton is not affected.

An Automaton for Intersection of Languages

Let us have the following two automata:





Do both of them accept the word ababb?

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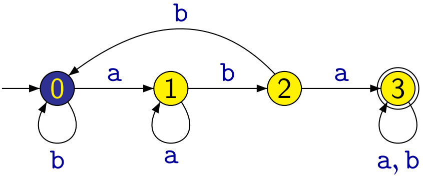

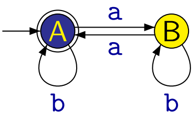

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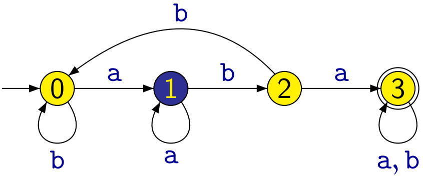

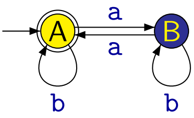

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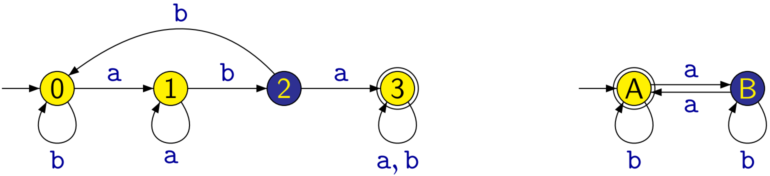

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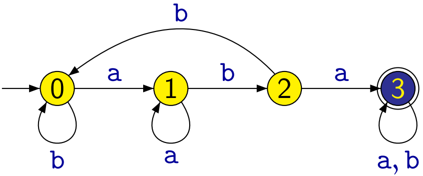

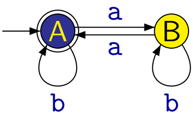

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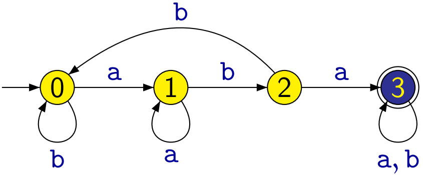

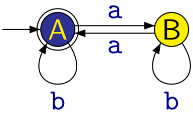

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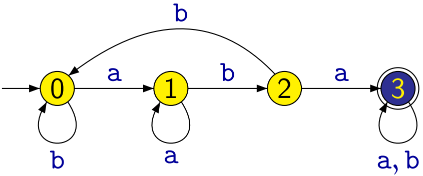

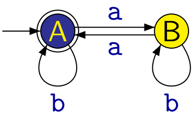

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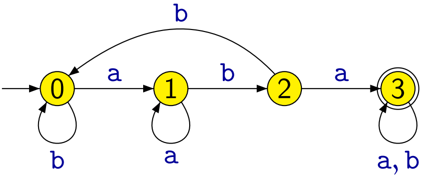

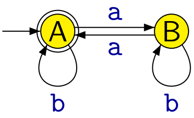

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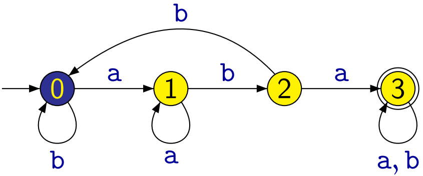

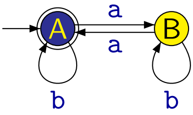

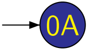

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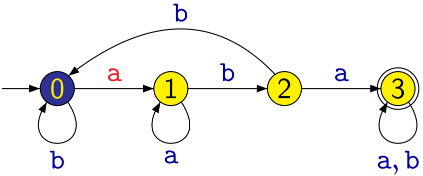

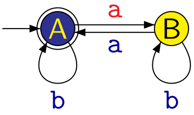

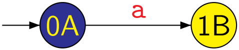

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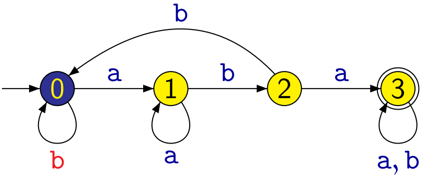

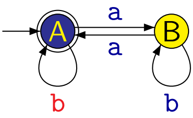

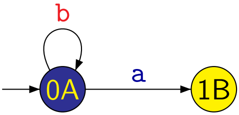

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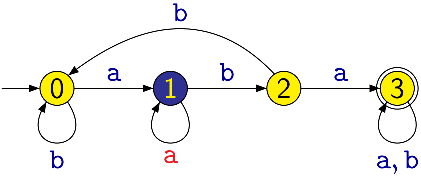

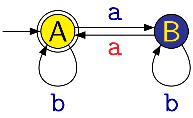

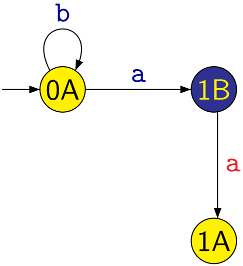

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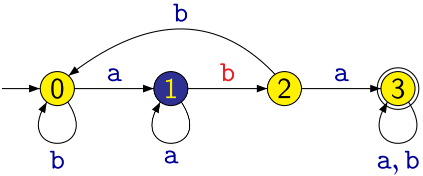

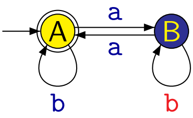

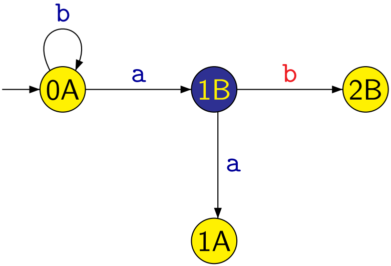

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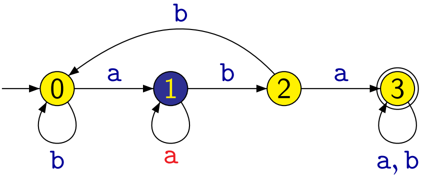

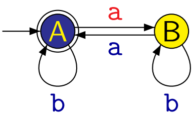

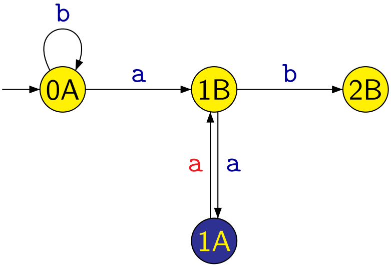

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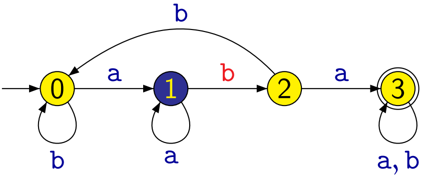

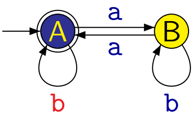

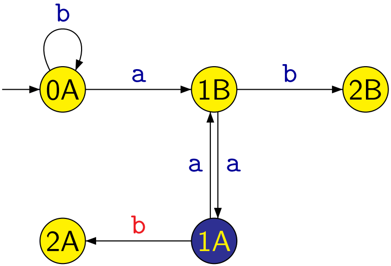

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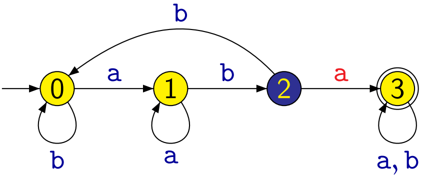

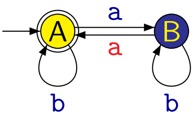

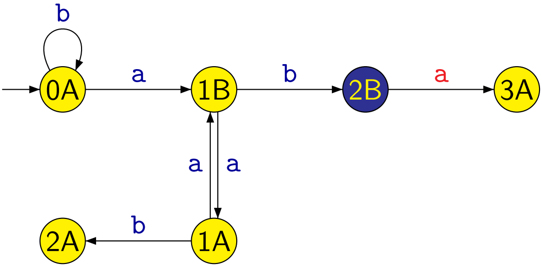

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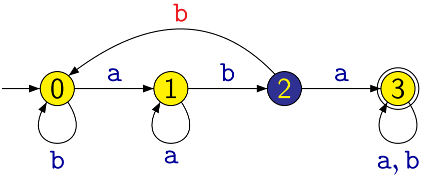

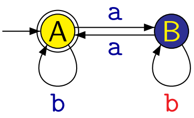

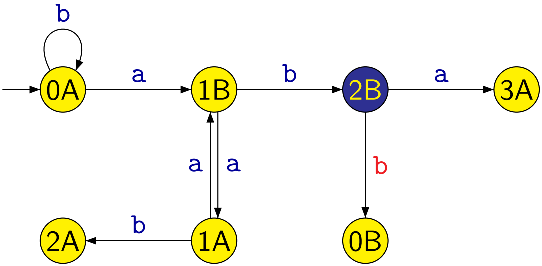

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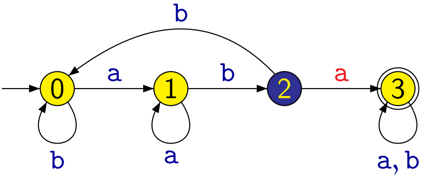

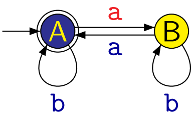

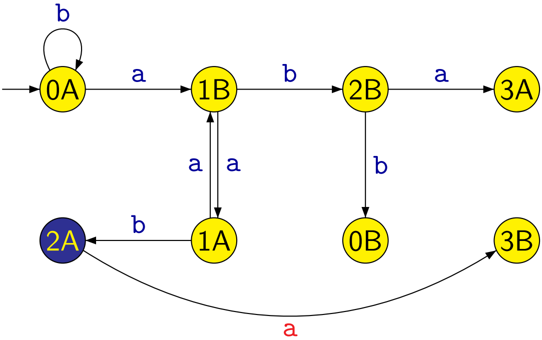

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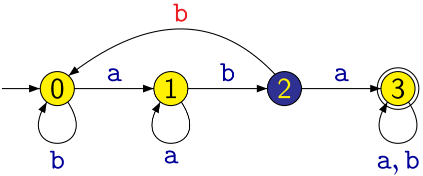

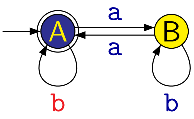

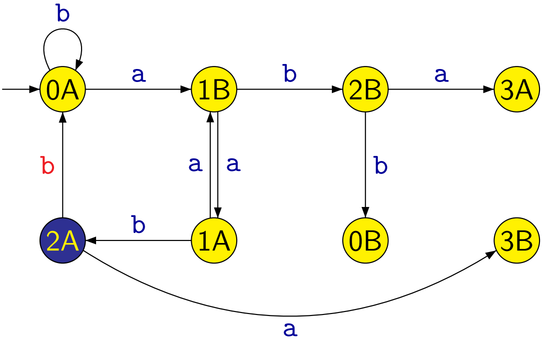

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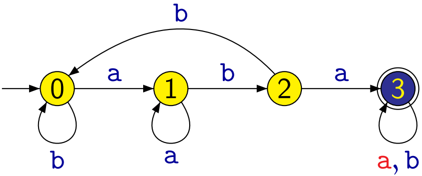

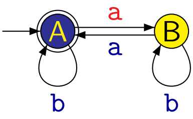

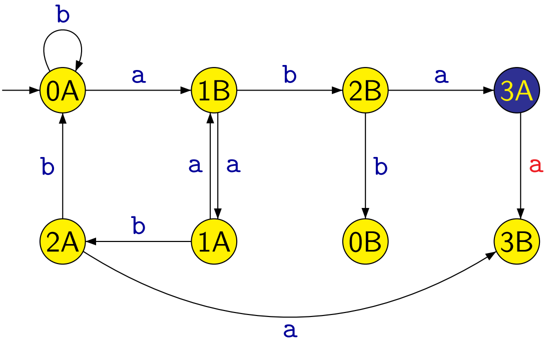

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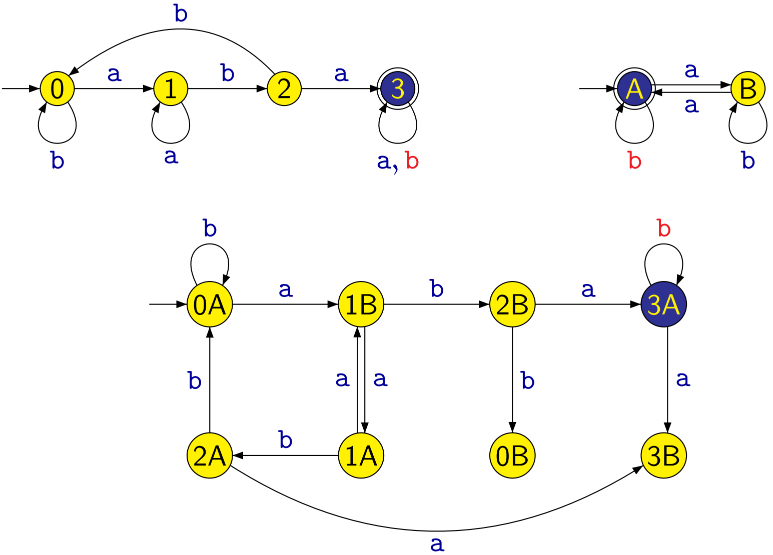

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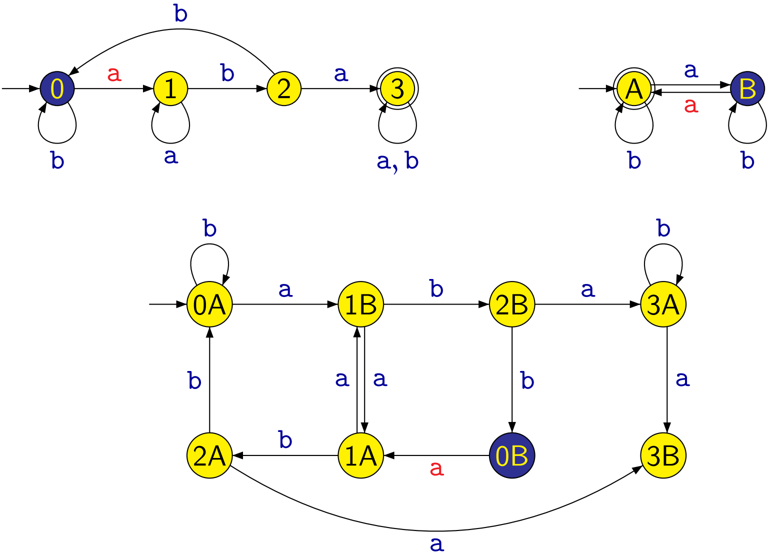

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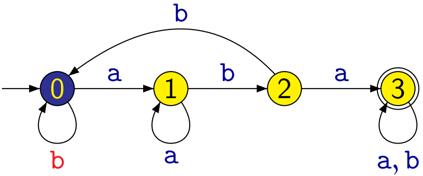

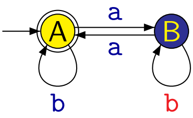

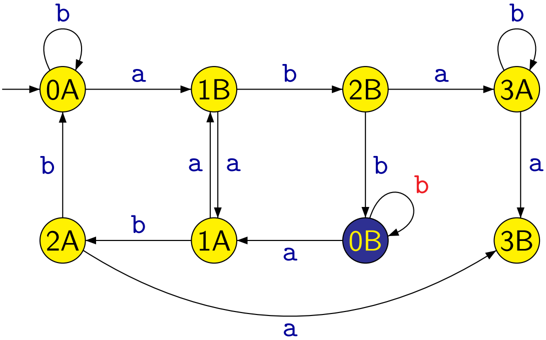

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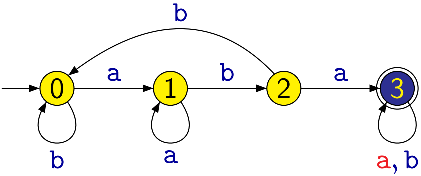

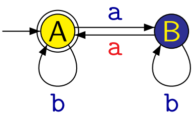

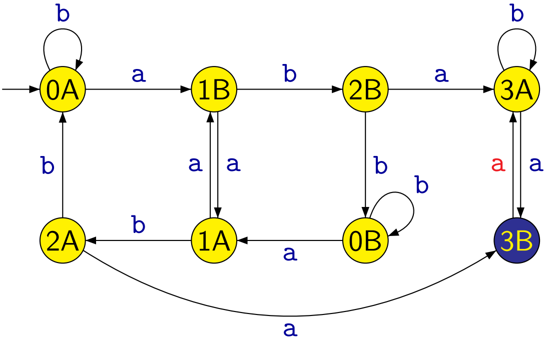

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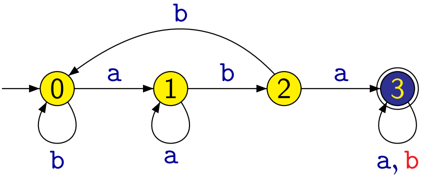
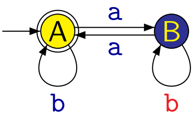
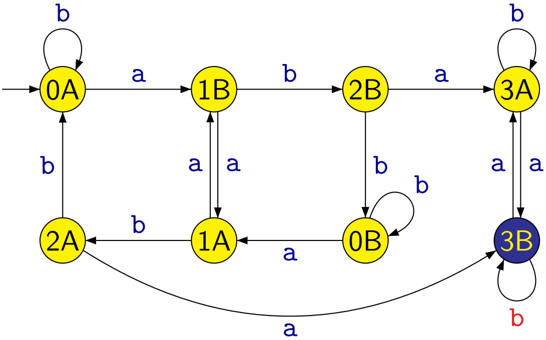
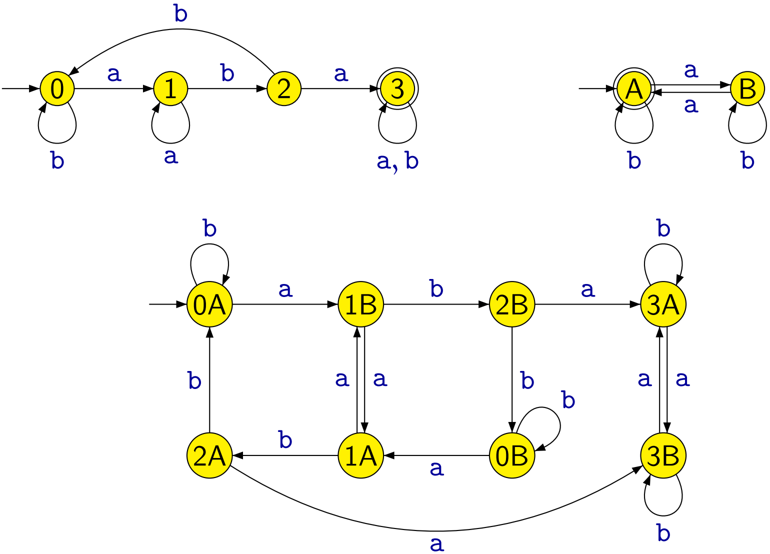


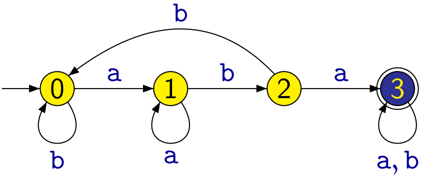


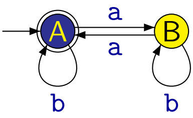


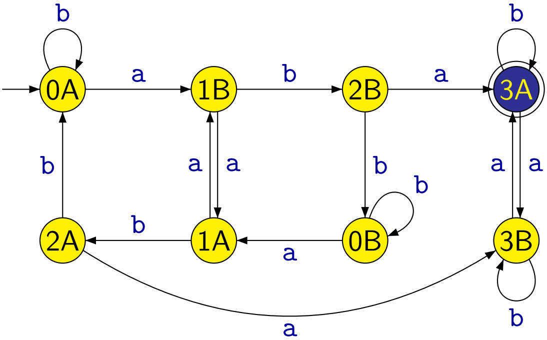
An Automaton for Intersection of Languages



An Automaton for Intersection of Languages







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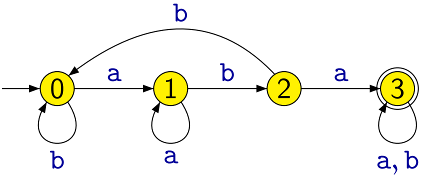
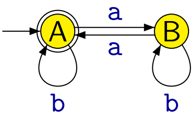
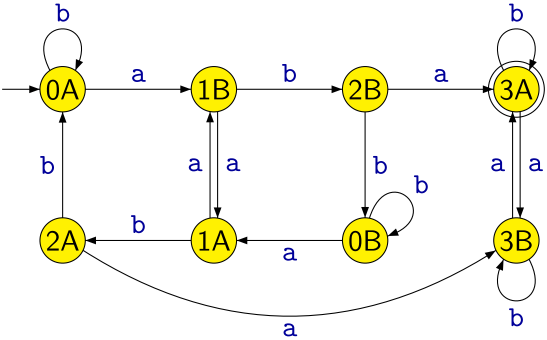






An Automaton for Intersection of Languages

Formally, the construction can be described as follows:

We assume we have two deterministic finite automata $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$.

We construct DFA $A = (Q, \Sigma, \delta, q_0, F)$ where:

- $Q = Q_1 \times Q_2$
- $\delta(\,(q_1, q_2), a\,) = (\,\delta_1(q_1, a), \delta_2(q_2, a)\,)$ for each $q_1 \in Q_1$, $q_2 \in Q_2$, $a \in \Sigma$
- $q_0 = (q_{01}, q_{02})$
- $F = F_1 \times F_2$

It is not difficult to check that for each word $w \in \Sigma^*$ we have $w \in L(A)$ iff $w \in L(A_1)$ and $w \in L(A_2)$, i.e.,

$$L(A) = L(A_1) \cap L(A_2)$$

Intersection of Regular Languages

Theorem

If languages $L_1, L_2 \subseteq \Sigma^*$ are regular then also the language $L_1 \cap L_2$ is regular.

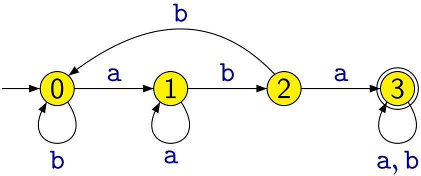
Proof: Let us assume that A_1 and A_2 are deterministic finite automata such that

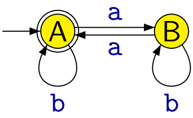
$$L_1 = L(A_1) \qquad\qquad L_2 = L(A_2)$$

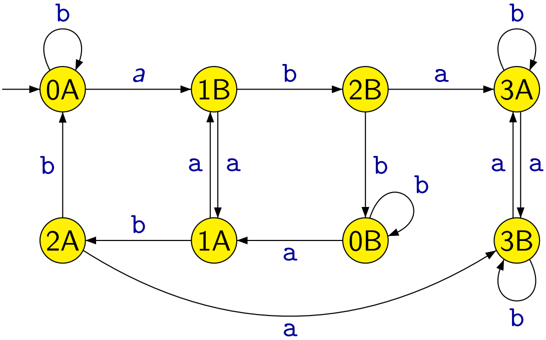
Using the described construction, we can construct a deterministic finite automaton A such that

$$L(A) = L(A_1) \cap L(A_2) = L_1 \cap L_2$$

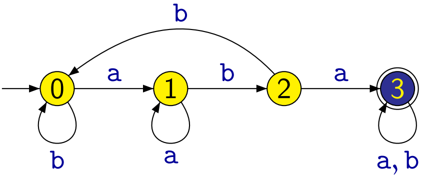
An Automaton for the Union of Languages

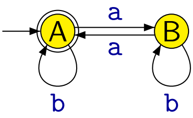


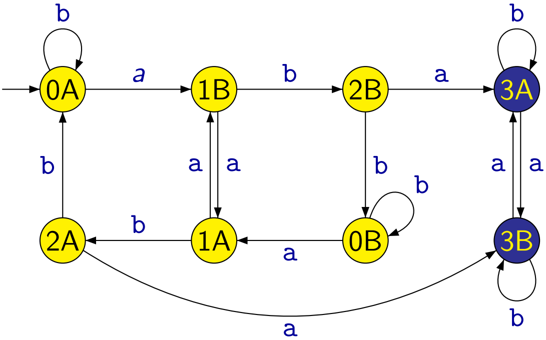




An Automaton for the Union of Languages







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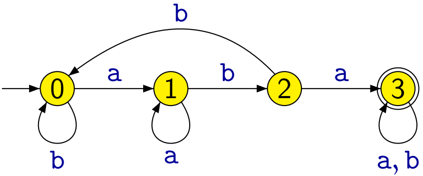
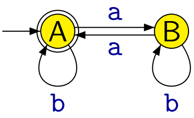
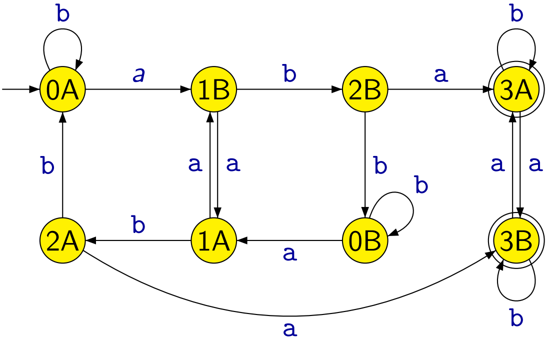
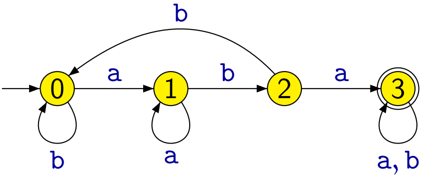
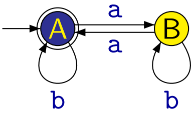


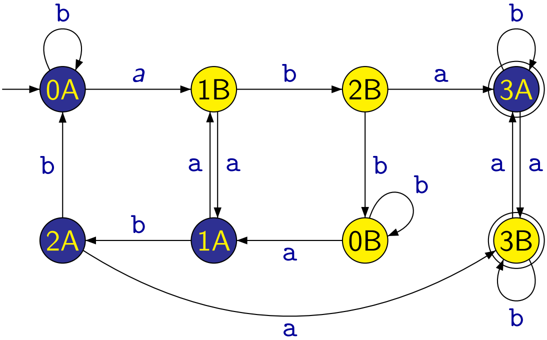




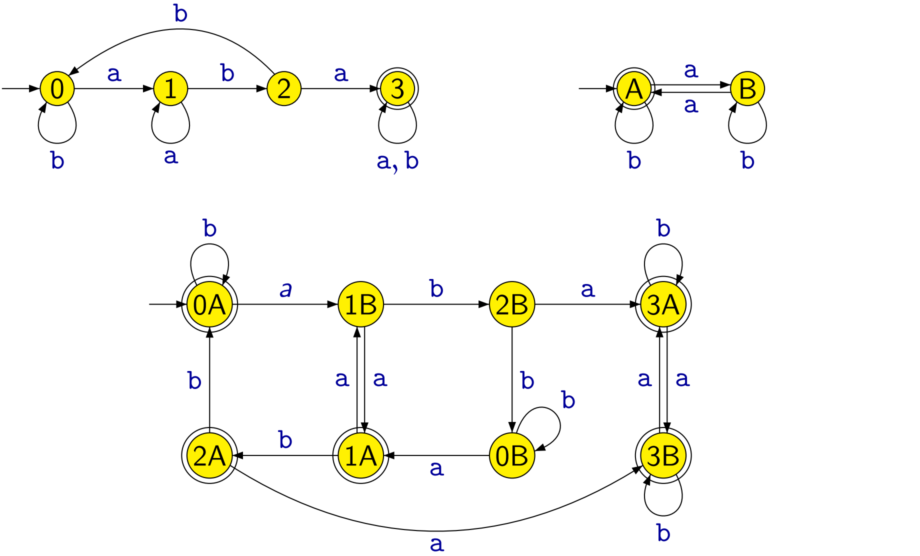
An Automaton for the Union of Languages







An Automaton for the Union of Languages



Union of Regular Languages

The construction of an automaton A that accepts the **union** of languages accepted by automata A_1 and A_2, i.e., the language

$$L(A_1) \cup L(A_1)$$

is almost identical as in the case of the automaton accepting $L(A_1) \cap L(A_2)$.

The only difference is the set of accepting states:

- $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$

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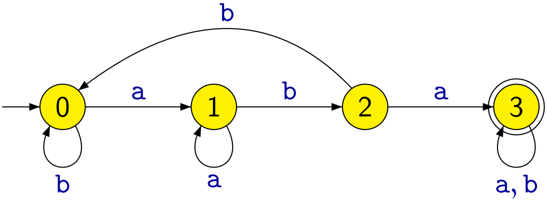
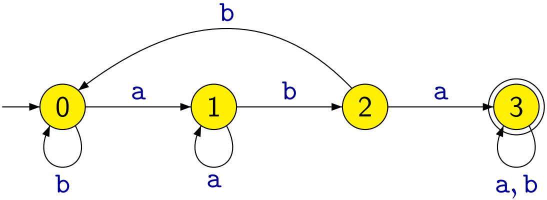
Theorem

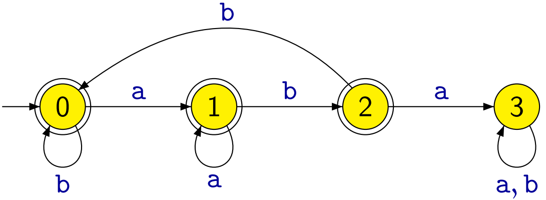
If languages $L_1, L_2 \subseteq \Sigma^*$ are regular then also the language $L_1 \cup L_2$ is regular.

An Automaton for the Complement of a Language



An Automaton for the Complement of a Language





Complement of a Regular Language

Given a DFA $A = (Q, \Sigma, \delta, q_0, F)$ we construct DFA $A' = (Q, \Sigma, \delta, q_0, Q - F)$.

It is obvious that for each word $w \in \Sigma^*$ we have $w \in L(A')$ iff $w \notin L(A)$, i.e.,

$$L(A') = \overline{L(A)}$$

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It is obvious that for each word $w \in \Sigma^*$ we have $w \in L(A')$ iff $w \notin L(A)$, i.e.,

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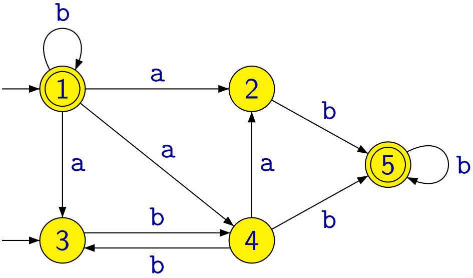
Theorem

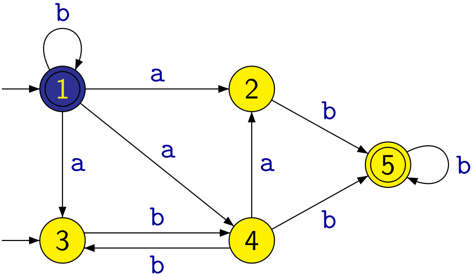
If a language L is regular then also its complement $\overline{L}$ is regular.

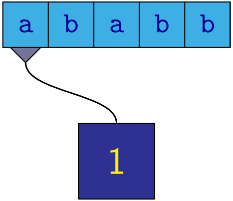
Nondeterministic Finite Automaton



- The number of transitions going from one state and labelled with the same symbol can be arbitrary (including zero).
- There can be more than one initial state in the automaton.

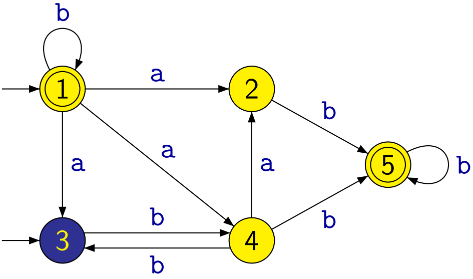
Nondeterministic Finite Automaton

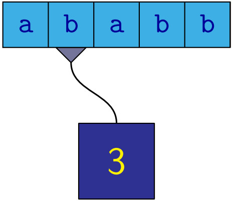




1

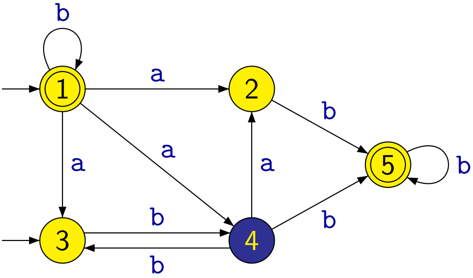
Nondeterministic Finite Automaton

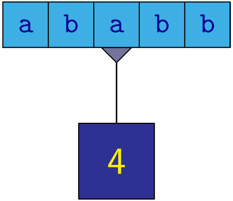




$$1 \xrightarrow{\mathtt{a}} 3$$

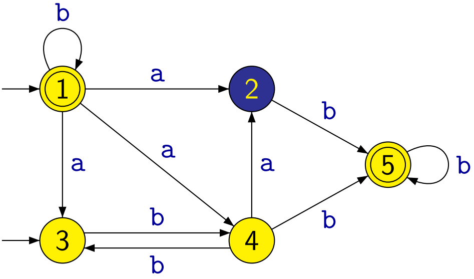
Nondeterministic Finite Automaton

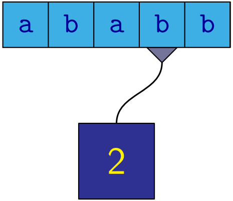




$$1 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 4$$

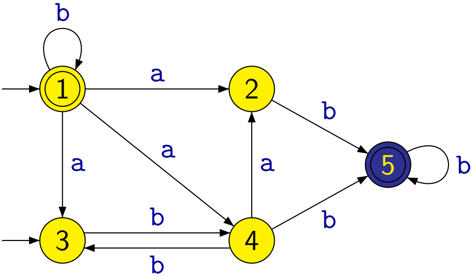
Nondeterministic Finite Automaton

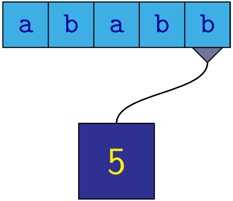




$$1 \xrightarrow{\mathtt{a}} 3 \xrightarrow{\mathtt{b}} 4 \xrightarrow{\mathtt{a}} 2$$

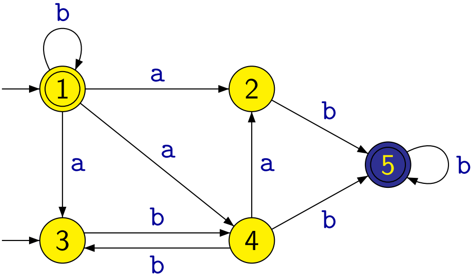
Nondeterministic Finite Automaton

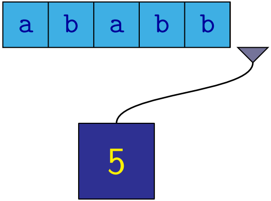




$$1 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{a} 2 \xrightarrow{b} 5$$

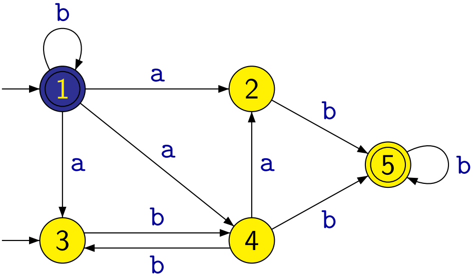
Nondeterministic Finite Automaton

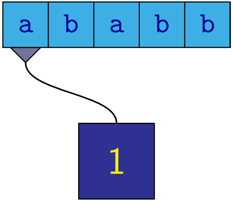




$$1 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{a} 2 \xrightarrow{b} 5 \xrightarrow{b} 5$$

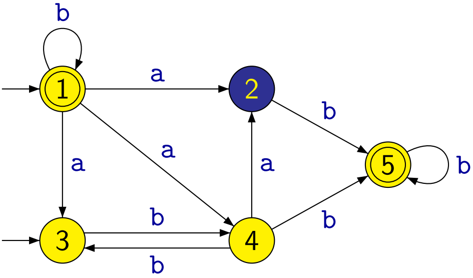
Nondeterministic Finite Automaton

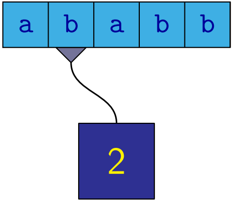




1

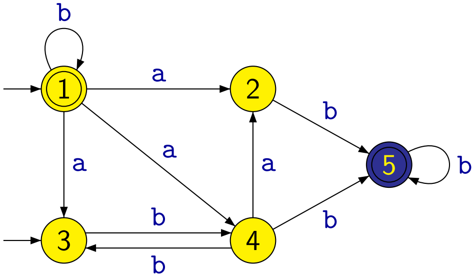
Nondeterministic Finite Automaton

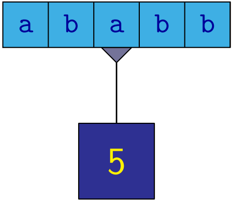




$$1 \xrightarrow{\mathtt{a}} 2$$

Nondeterministic Finite Automaton

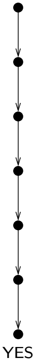


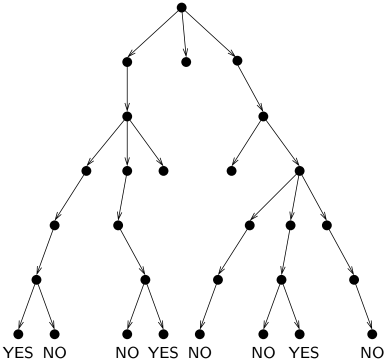


$$1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{b}} 5$$

Nondeterministic Finite Automaton

A nondeterministic finite automaton accepts a given word if there **exists** at least one computation of the automaton that accepts the word.





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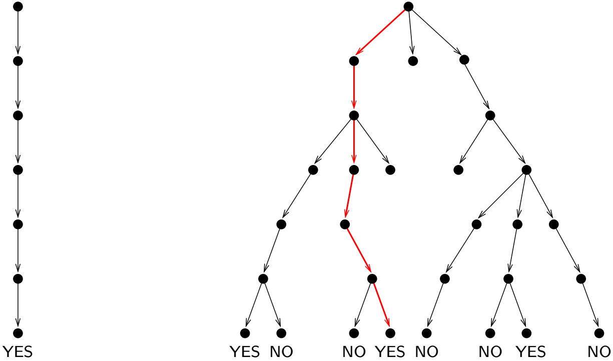
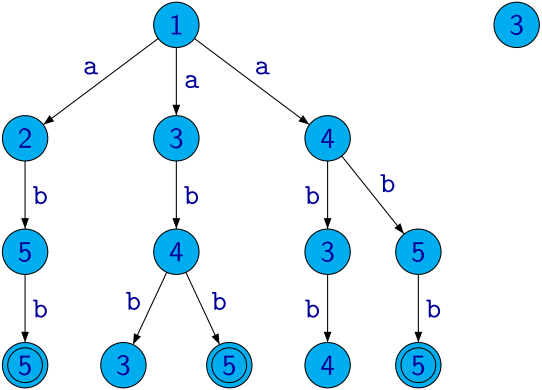


Nondeterministic Finite Automaton

	a	b
↔ 1	2, 3, 4	1
2	–	5
→ 3	–	4
4	2	3, 5
← 5	–	5



Example: A forest representing all possible computations over the word `abb`.

Nondeterministic Finite Automaton

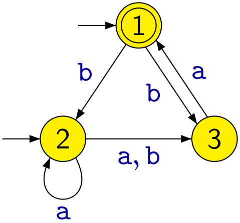
Formally, a **nondeterministic finite automaton** (**NFA**) is defined as a tuple

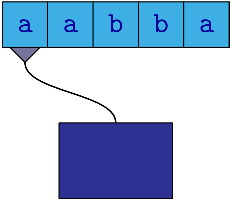
$$(Q, \Sigma, \delta, I, F)$$

where:

- Q is a finite set of **states**
- Σ is a finite **alphabet**
- $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$ is a **transition fuction**
- $I \subseteq Q$ is a set of **initial states**
- $F \subseteq Q$ is a set of **accepting states**

Transformation of NFA to DFA





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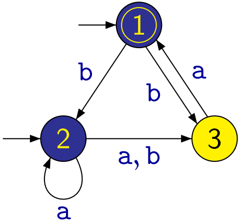

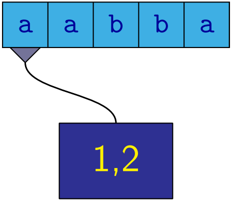

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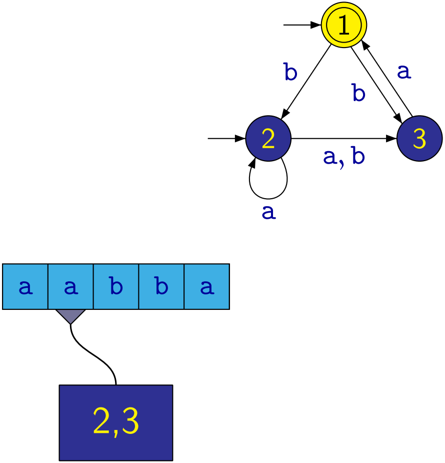

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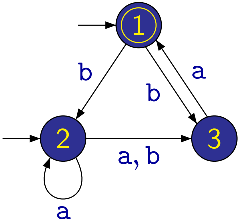

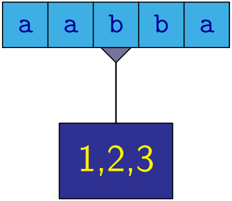

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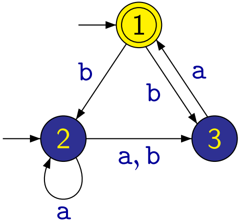
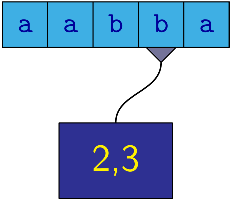
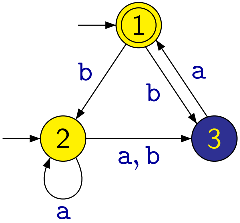
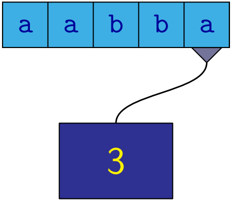




a	a	b	b	a

Transformation of NFA to DFA





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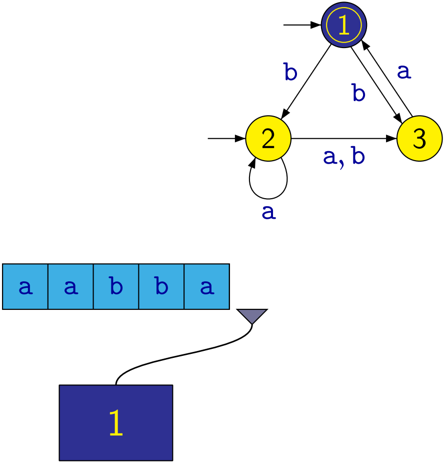

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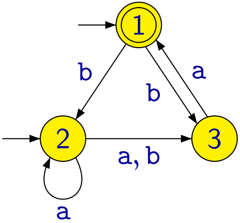
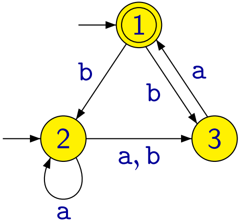
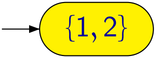


Transformation of NFA to DFA





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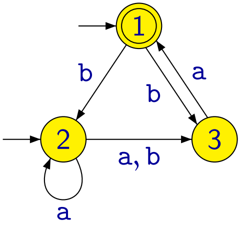

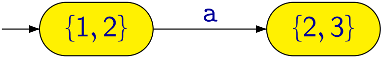

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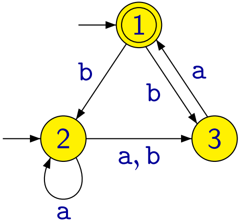

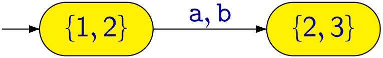

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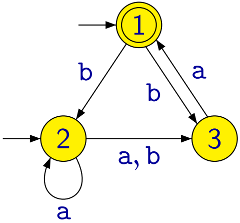

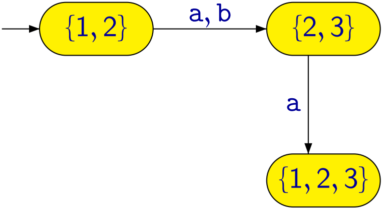

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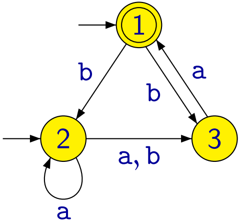

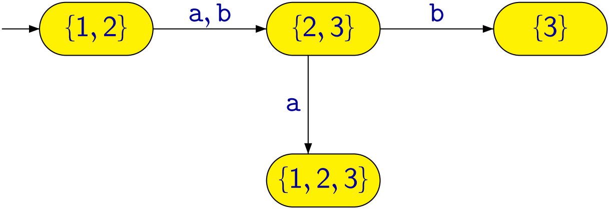

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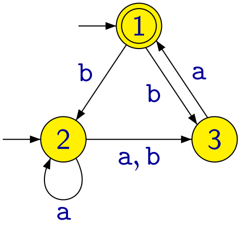

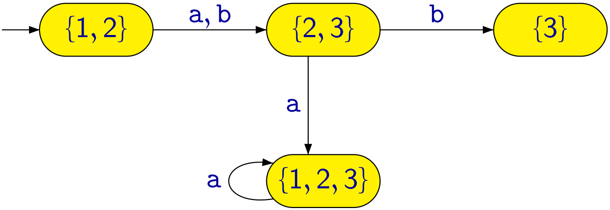

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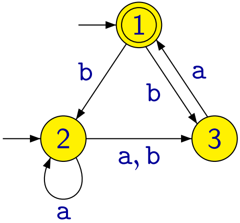

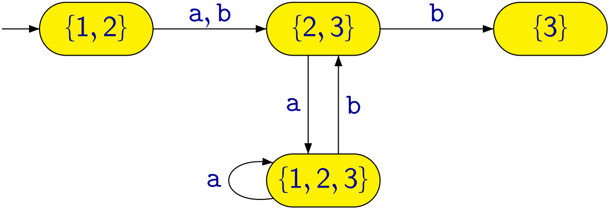

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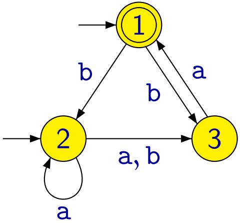

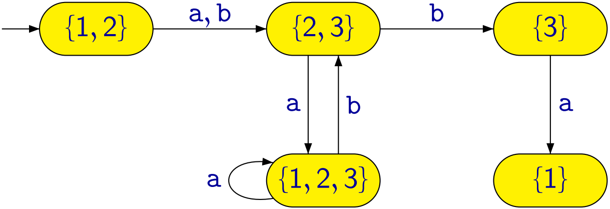

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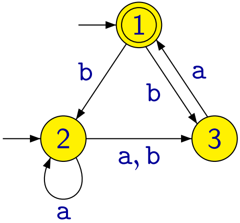

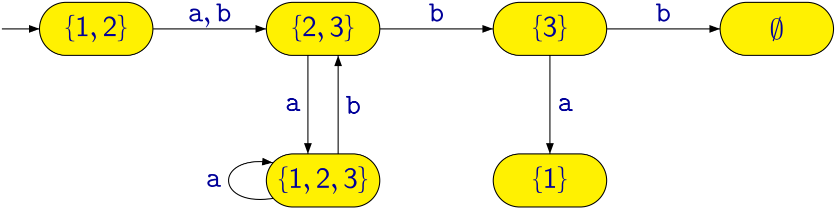

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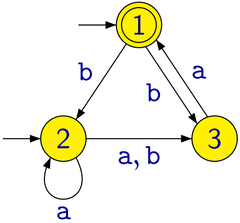
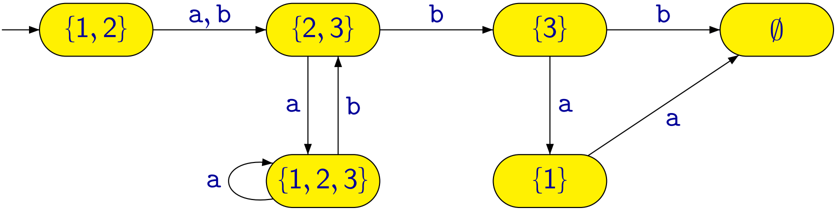
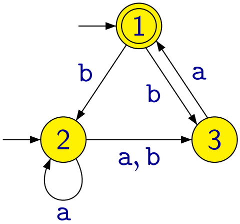
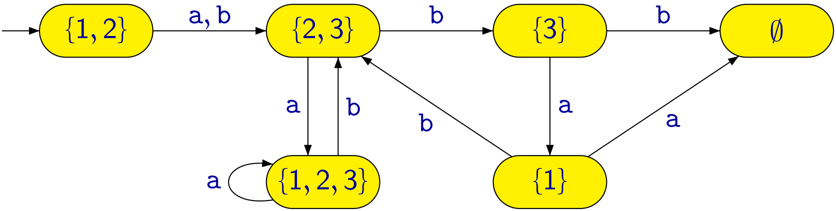




Transformation of NFA to DFA





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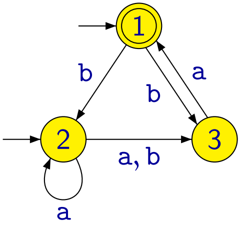
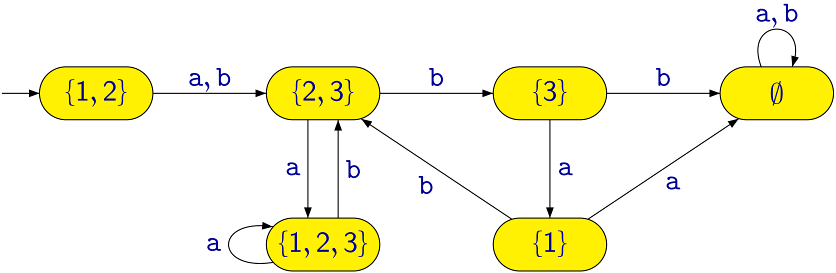
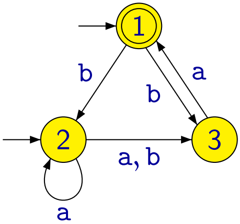
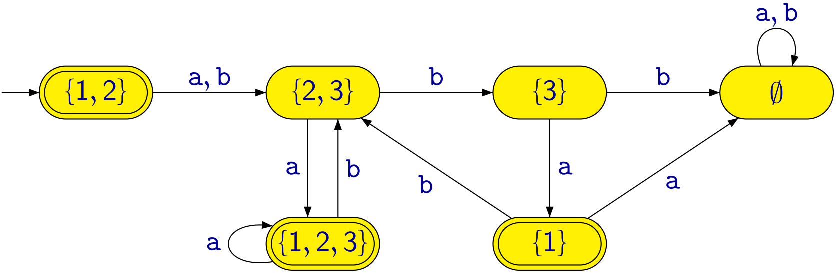




Transformation of NFA to DFA





Transformation of NFA to DFA

	a	b
↔ 1	—	2, 3
→ 2	2, 3	3
3	1	—

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	—

	a	b

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	

Transformation of NFA to DFA

	a	b
↔ 1	—	2, 3
→ 2	2, 3	3
3	1	—

	a	b
↔ {1, 2}	{2, 3}	
{2, 3}		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	–	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	–

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	
$\leftarrow \{1, 2, 3\}$		

Transformation of NFA to DFA

	a	b
↔ 1	—	2, 3
→ 2	2, 3	3
3	1	—

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\leftarrow \{1,2,3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$		
$\{3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	
$\{3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\leftarrow \{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$
$\{3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	
$\leftarrow \{1\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	—

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\leftarrow \{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$		
$\emptyset$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$	$\emptyset$	
$\emptyset$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$	$\emptyset$	$\{2, 3\}$
$\emptyset$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$	$\emptyset$	$\{2, 3\}$
$\emptyset$	$\emptyset$	$\emptyset$

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	—	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	—

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$	$\emptyset$	$\{2, 3\}$
$\emptyset$	$\emptyset$	$\emptyset$

	a	b
$\leftrightarrow 1$	2	2
2	3	4
$\leftarrow 3$	3	2
4	5	6
$\leftarrow 5$	6	2
6	6	6

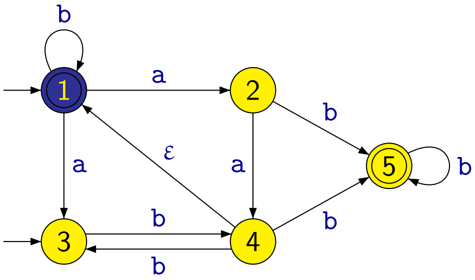
Transformation of NFA to DFA

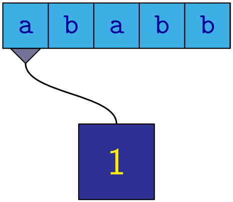
Remark: When a nondeterministic automaton with n states is transformed into a deterministic one, the resulting automaton can have 2^n states.

For example when we transform an automaton with 20 states, the resulting automaton can have $2^{20} = 1048576$ states.

It is often the case that the resulting automaton has far less than 2^n states. However, the worst cases are possible.

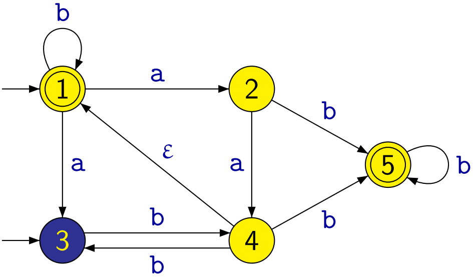
Generalized Nondeterministic Finite Automaton

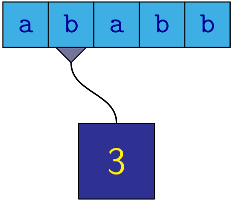




1

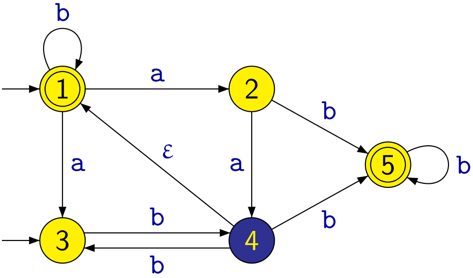
Generalized Nondeterministic Finite Automaton

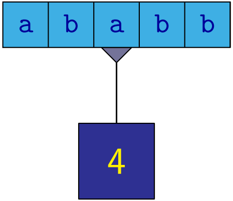




$1 \xrightarrow{a} 3$

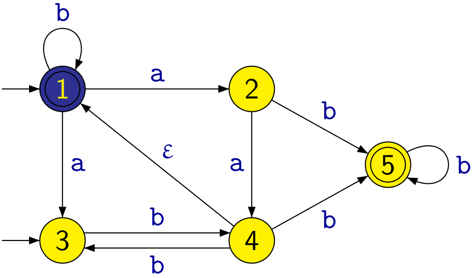
Generalized Nondeterministic Finite Automaton

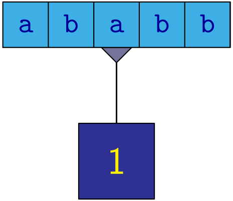




$$1 \xrightarrow{a} 3 \xrightarrow{b} 4$$

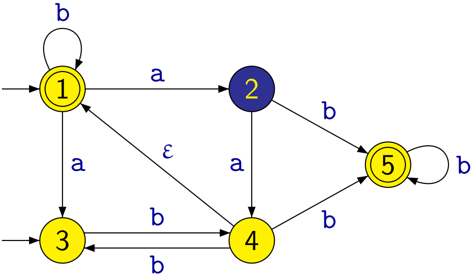
Generalized Nondeterministic Finite Automaton

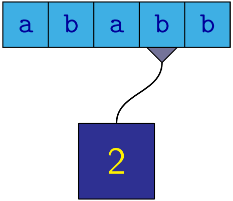




$1 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{\varepsilon} 1$

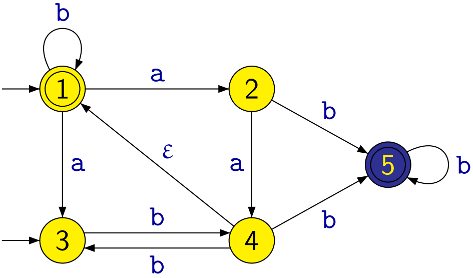
Generalized Nondeterministic Finite Automaton

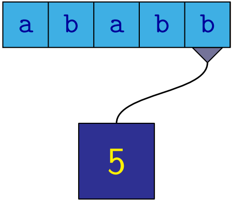




$$1 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{\varepsilon} 1 \xrightarrow{a} 2$$

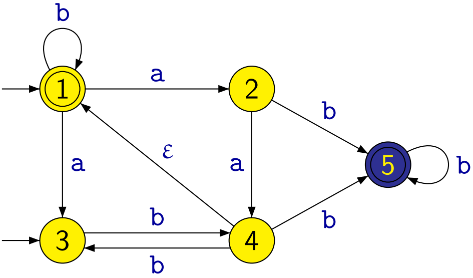
Generalized Nondeterministic Finite Automaton

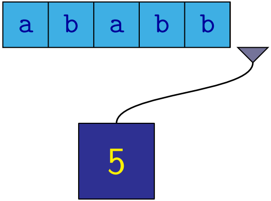




$$1 \xrightarrow{\mathtt{a}} 3 \xrightarrow{\mathtt{b}} 4 \xrightarrow{\varepsilon} 1 \xrightarrow{\mathtt{a}} 2 \xrightarrow{\mathtt{b}} 5$$

Generalized Nondeterministic Finite Automaton





$1 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{\varepsilon} 1 \xrightarrow{a} 2 \xrightarrow{b} 5 \xrightarrow{b} 5$

Generalized Nondeterministic Finite Automaton

Compared to a nondeterministic finite automaton, a **generalized nondeterministic finite automaton** has the so called ε-**transitions**, i.e., transitions labelled with symbol ε.

When ε-transition is performed, only the state of the control unit is changed but the head on the tape is not moved.

Remark: The computations of a generalized nondeterministic automaton can be of an arbitrary length, even infinite (if the graph of the automaton contains a cycle consisting only of ε-transitions) regardless of the length of the word on the tape.

Generalized Nondeterministic Finite Automaton

Formally, a **generalized nondeterministic finite automaton** (**GNFA**) is defined as a tuple

$$(Q, \Sigma, \delta, I, F)$$

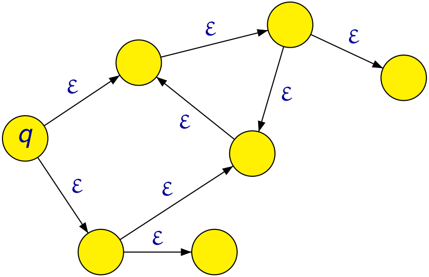
where:

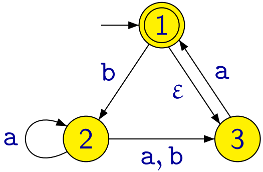
- Q is a finite set of **states**
- Σ is a finite **alphabet**
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q)$ is a **transition function**
- $I \subseteq Q$ is a set of **initial states**
- $F \subseteq Q$ is a set of **accepting states**

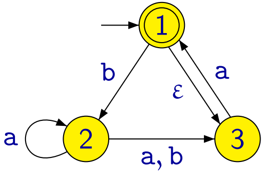
Remark: NFA can be viewed as a special case of GNFA, where $\delta(q, \varepsilon) = \emptyset$ for all $q \in Q$.

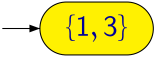
Transformation to a Deterministic Finite Automaton

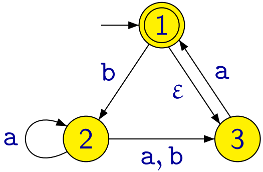
A generalized nondeterministic finite automaton can be transformed into a deterministic one using a similar construction as a nondeterministic finite automaton with the difference that we add to sets of states also all states that are reachable from already added states by some sequence of ε-transitions.




1
2
3
b
a
ε
a
a, b


1
b
ε
a
a
2
3
a, b


{1, 3}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a
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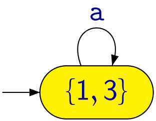
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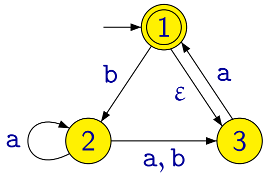
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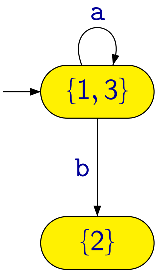
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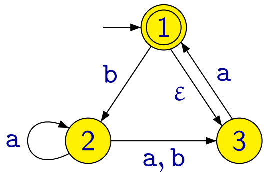
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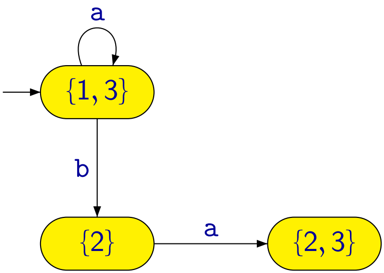
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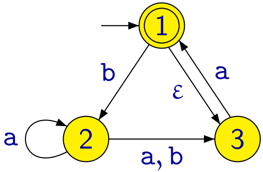
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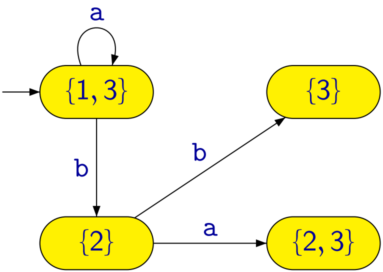
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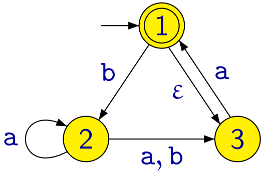
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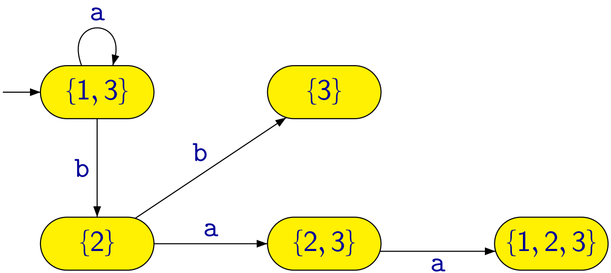
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$\{2,3\}$
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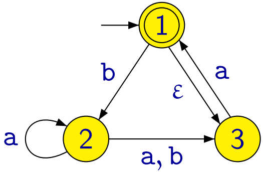
$\{1,2,3\}$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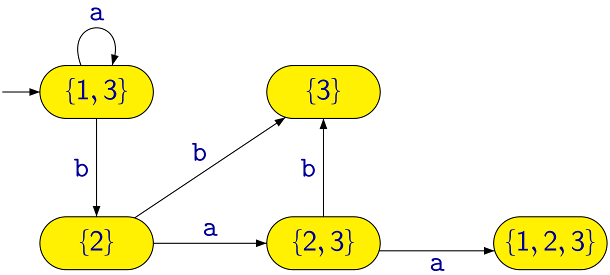
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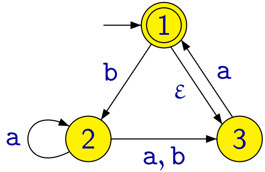
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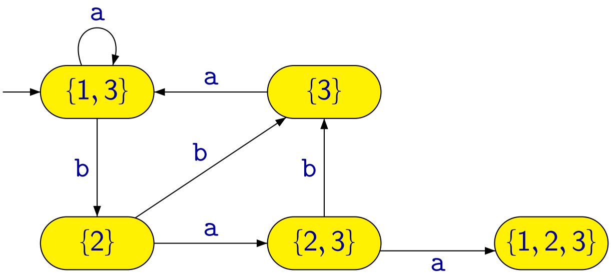
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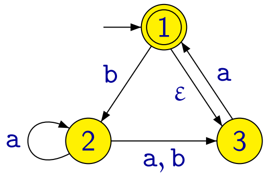
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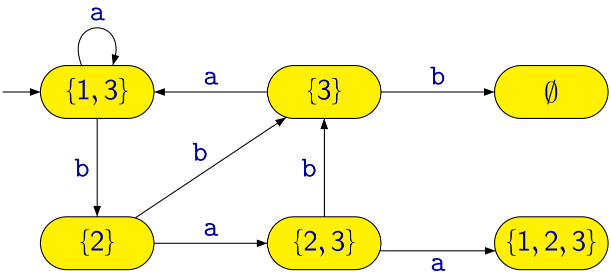
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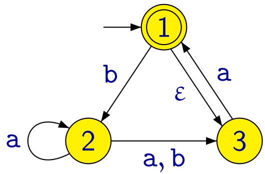
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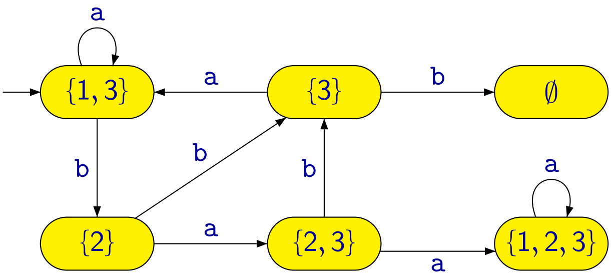
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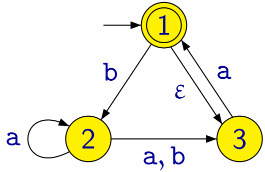
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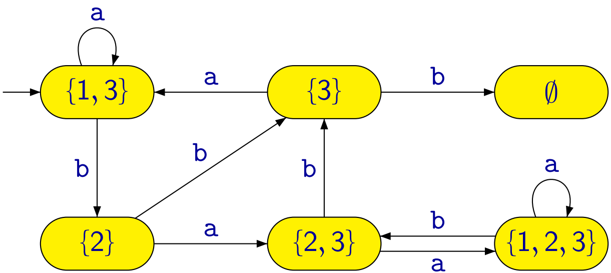
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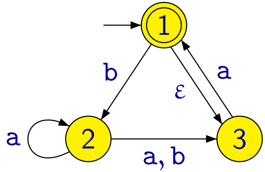
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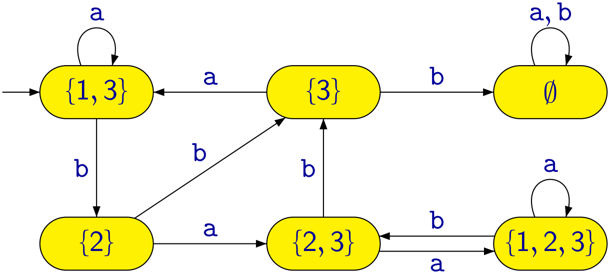
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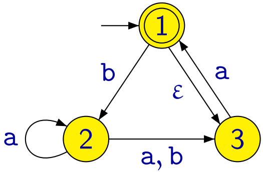
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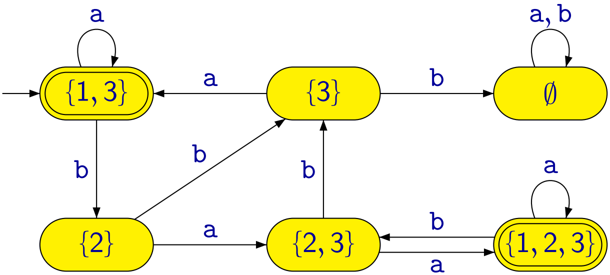
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{3}
b
∅
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b
b
a
{2}
a
{2, 3}
b
{1, 2, 3}
a

Transformation of GNFA to DFA

Before formally describing the transition of GNFA to DFA, let us introduce some auxiliary definitions.

Let us assume some given GNFA $A = (Q, \Sigma, \delta, I, F)$.

Let us define the function $\hat{\delta} : \mathcal{P}(Q) \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q)$ so that for $K \subseteq Q$ and $a \in \Sigma \cup \{\varepsilon\}$ there is

$$\hat{\delta}(K, a) = \bigcup_{q \in K} \delta(q, a)$$

Transformation of GNFA to DFA

For $K \subseteq Q$, let $Cl_\varepsilon(K)$ be the all states reachable from the states from the set K by some arbitrary sequence of ε-transitions.

This means that the function $Cl_\varepsilon : \mathcal{P}(Q) \to \mathcal{P}(Q)$ is defined so that for $K \subseteq Q$ is $Cl_\varepsilon(K)$ the smallest (with respect to inclusion) set satisfying the following two conditions:

- $K \subseteq Cl_\varepsilon(K)$
- For each $q \in Cl_\varepsilon(K)$ it holds that $\delta(q, \varepsilon) \subseteq Cl_\varepsilon(K)$.

Remark: Let us note that $Cl_\varepsilon(Cl_\varepsilon(K)) = Cl_\varepsilon(K)$ for arbitrary K.

Let us also note that in the case of NFA (where $\delta(q, \varepsilon) = \emptyset$ for each $q \in Q$) is $Cl_\varepsilon(K) = K$.

Transformation of GNFA to DFA

For a given GNFA $A = (Q, \Sigma, \delta, I, F)$ we can now construct DFA $A' = (Q', \Sigma, \delta', q_0', F')$, where:

- $Q' = \mathcal{P}(Q)$ (so $K \in Q'$ means that $K \subseteq Q$)
- $\delta' : Q' \times \Sigma \to Q'$ is defined so that for $K \in Q'$ and $a \in \Sigma$:

$$\delta'(K, a) = Cl_\varepsilon(\hat{\delta}(Cl_\varepsilon(K), a))$$

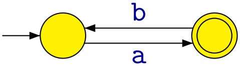
- $q_0' = Cl_\varepsilon(I)$
- $F' = \{K \in Q' \mid Cl_\varepsilon(K) \cap F \neq \emptyset\}$

It is not difficult to verify that $L(A) = L(A')$.

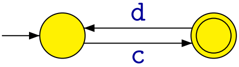
Concatenation of Languages

$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

A_1:

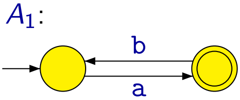


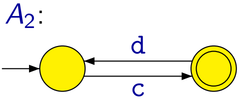
A_2:

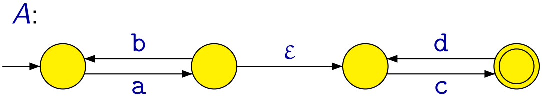


Concatenation of Languages

$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$



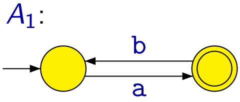


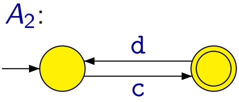


$L(A) = L(A_1) \cdot L(A_2)$

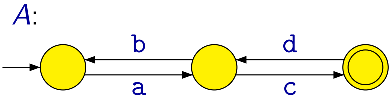
Concatenation of Languages

$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$



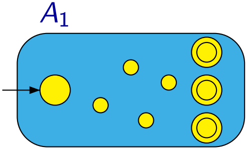


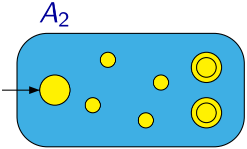
An incorrect construction:



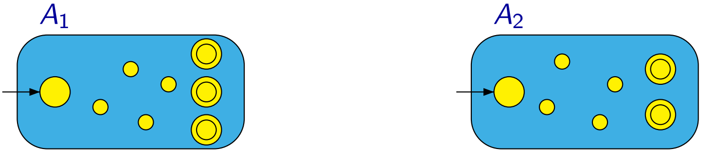
$\mathtt{acdbac} \in L(A)$ but $\mathtt{acdbac} \notin L(A_1) \cdot L(A_2)$

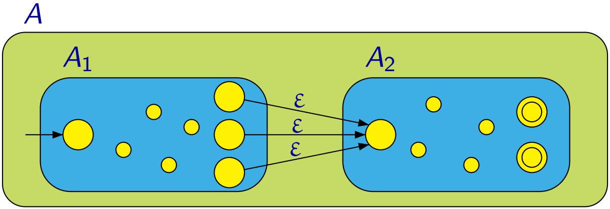
Concatenation of Languages



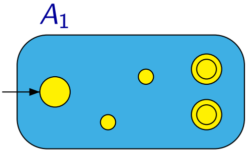


Concatenation of Languages

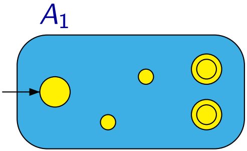


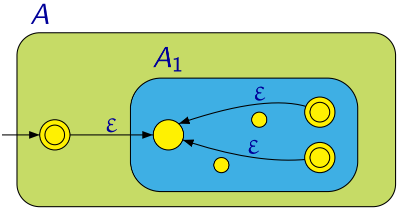


Iteration of a Language



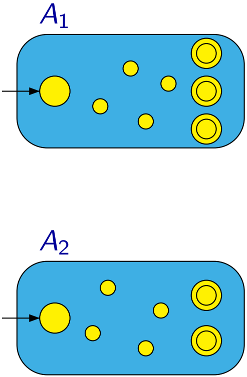
Iteration of a Language





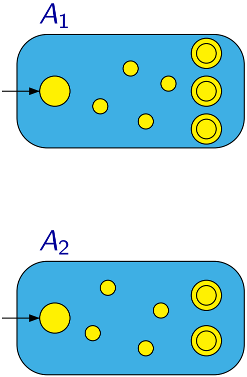
Union of Languages

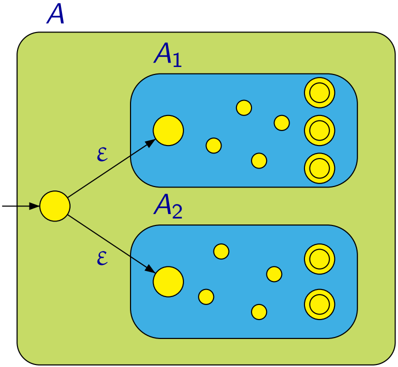
An alternative construction for the union of languages:



Union of Languages

An alternative construction for the union of languages:





Closure Properties of the Class of Regular Languages

The set of (all) regular languages is closed with respect to:

- union
- intersection
- complement
- concatenation
- iteration
- ...

Transformation of a Regular Expression to a Finite Automaton

Proposition

Every language that can be represented by a regular expression is regular (i.e., it is accepted by some finite automaton).

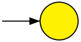
Proof: It is sufficient to show how to construct for a given regular expression α a finite automaton accepting the language $[\alpha]$.

The construction is recursive and proceeds by the structure of the expression α:

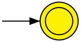
- If α is a elementary expression (i.e., $\emptyset$, ε or a):
  - We construct the corresponding automaton directly.
- If α is of the form $(\beta + \gamma)$, $(\beta \cdot \gamma)$ or (β^*):
  - We construct automata accepting languages $[\beta]$ and $[\gamma]$ recursively.
  - Using these two automata, we construct the automaton accepting the language $[\alpha]$.

Transformation of a Regular Expression to a Finite Automaton

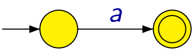
The automata for the elementary expressions:



$\emptyset$



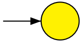
ε



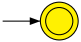
a

Transformation of a Regular Expression to a Finite Automaton

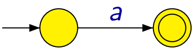
The automata for the elementary expressions:



$\emptyset$

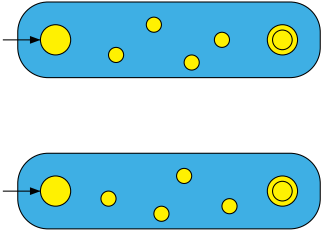


ε



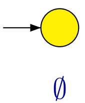
a

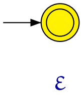
The construction for the union:

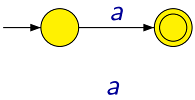


Transformation of a Regular Expression to a Finite Automaton

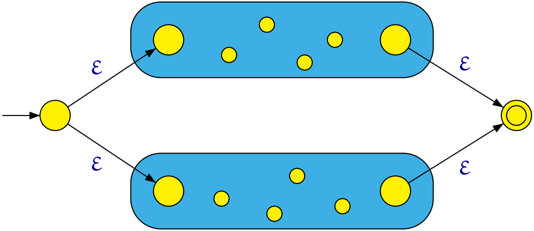
The automata for the elementary expressions:





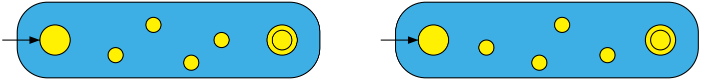


The construction for the union:



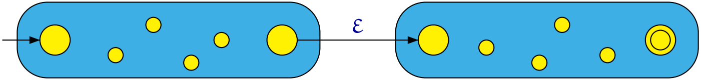
Transformation of a Regular Expression to a Finite Automaton

The construction for the concatenation:



Transformation of a Regular Expression to a Finite Automaton

The construction for the concatenation:



Transformation of a Regular Expression to a Finite Automaton

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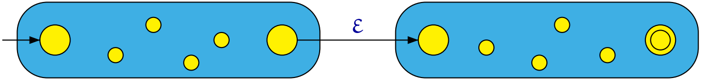
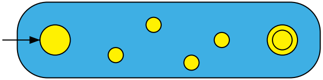
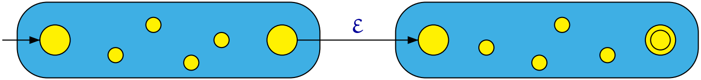


The construction for the iteration:

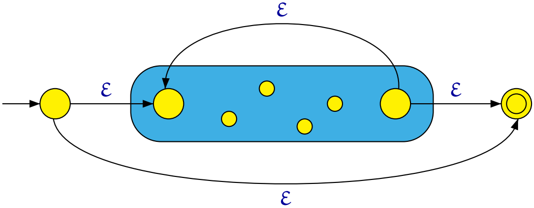


Transformation of a Regular Expression to a Finite Automaton

The construction for the concatenation:



The construction for the iteration:

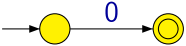


Transformation of a Regular Expression to a Finite Automaton

Example: The construction of an automaton for expression $((0+1)\cdot 1)^*$:

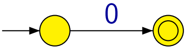
Transformation of a Regular Expression to a Finite Automaton

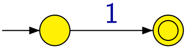
Example: The construction of an automaton for expression $((0 + 1) \cdot 1)^*$:



Transformation of a Regular Expression to a Finite Automaton

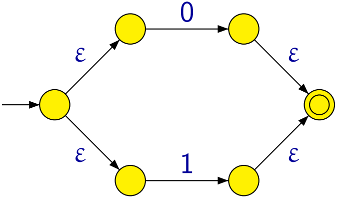
Example: The construction of an automaton for expression $((0 + 1) \cdot 1)^*$:





Transformation of a Regular Expression to a Finite Automaton

Example: The construction of an automaton for expression $((0+1)\cdot 1)^*$:



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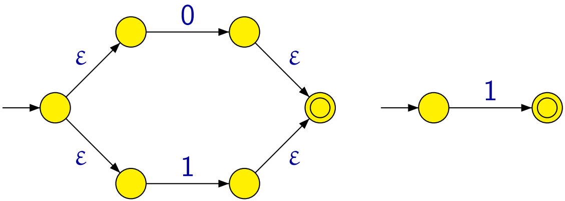

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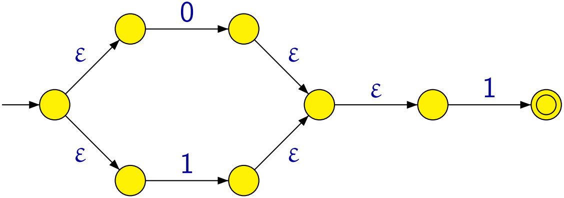

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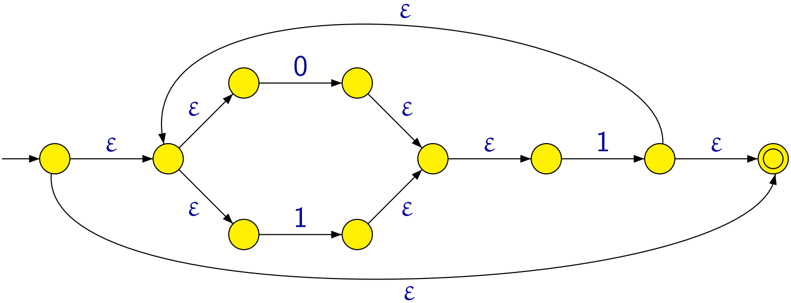


Transformation of a Regular Expression to a Finite Automaton

If an expression α consists of n symbols (not counting parenthesis) then the resulting automaton has:

- at most $2n$ states,
- at most $4n$ transitions.

Remark: By transforming the generalized nondeterministic automaton into a deterministic one, the number of states can grow exponentially, i.e., the resulting automaton can have up to $2^{2n} = 4^n$ states.

Transformation of an Automaton to a Regular Expression

Proposition

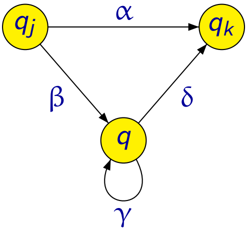
Every regular language can be represented by some regular expression.

Proof: It is sufficient to show how to construct for a given finite automaton A a regular expression α such that $[\alpha] = L(A)$.

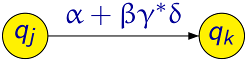
- We modify A in such a way that ensures it has exactly one initial and exactly one accepting state.
- Its states will be removed one by one.
- Its transitions will be labelled with regular expressions.
- The resulting automaton will have only two states – the initial and the accepting, and only one transition labelled with the resulting regular expression.

Transformation of an Automaton to a Regular Expression

The main idea: If a state q is removed, for every pair of remaining states q_j, q_k we extend the label on a transition from q_j to q_k by a regular expression representing paths from q_j to q_k going through q.



After removing of the state q:



Transformation of an Automaton to a Regular Expression

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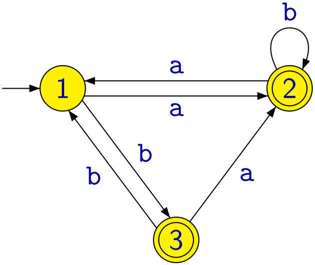

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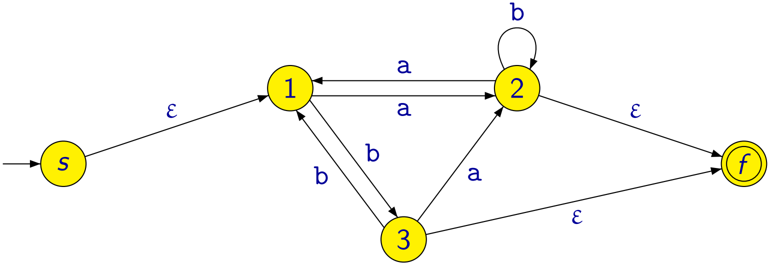

Transformation of an Automaton to a Regular Expression

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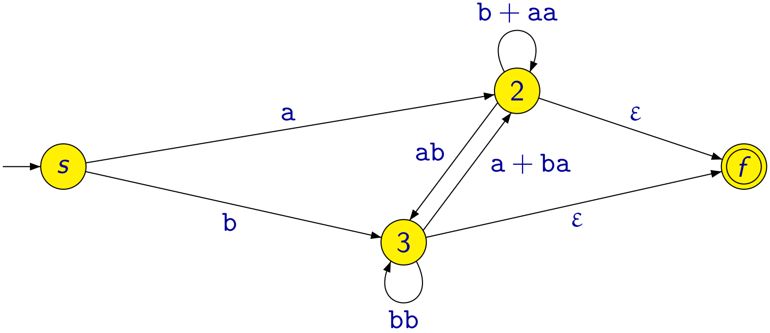

Transformation of an Automaton to a Regular Expression

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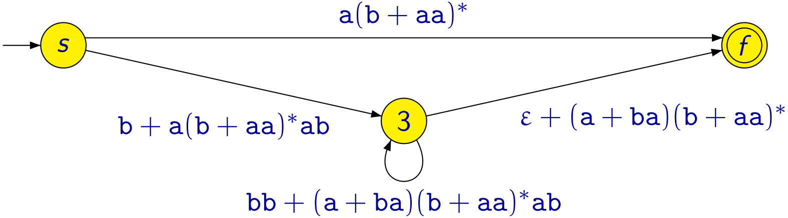
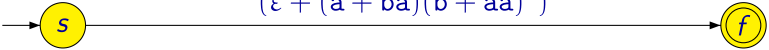

$\mathtt{a(b+aa)^*}$
s
f
3
$\mathtt{b+a(b+aa)^*ab}$
$\varepsilon + \mathtt{(a+ba)(b+aa)^*}$
$\mathtt{bb+(a+ba)(b+aa)^*ab}$

Transformation of an Automaton to a Regular Expression

Example:



$$\mathtt{a}(\mathtt{b}+\mathtt{aa})^*+\\(\mathtt{b}+\mathtt{a}(\mathtt{b}+\mathtt{aa})^*\mathtt{ab})\\(\mathtt{bb}+(\mathtt{a}+\mathtt{ba})(\mathtt{b}+\mathtt{aa})^*\mathtt{ab})^*\\(\varepsilon+(\mathtt{a}+\mathtt{ba})(\mathtt{b}+\mathtt{aa})^*)$$

Equivalence of Finite Automata and Regular Expressions

Theorem

A language is regular iff it can be represented by a regular expression.

Context-Free Grammars

Context-Free Grammars

Example: We would like to describe a language of arithmetic expressions, containing expressions such as:

```
175     (9+15)     (((10-4)*((1+34)+2))/(3+(-37)))
```

For simplicity we assume that:

- Expressions are fully parenthesized.
- The only arithmetic operations are "+", "-", "*", "/"and unary "-".
- Values of operands are natural numbers written in decimal — a number is represented as a non-empty sequence of digits.

Alphabet: $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, -, *, /, (,)\}$

Context-Free Grammars

Example (cont.): A description by an inductive definition:

- **Digit** is any of characters `0`, `1`, `2`, `3`, `4`, `5`, `6`, `7`, `8`, `9`.
- **Number** is a non-empty sequence of digits, i.e.:
  - If α is a digit then α is a number.
  - If α is a digit and β is a number then also $\alpha\beta$ is a number.
- **Expression** is a sequence of symbols constructed according to the following rules:
  - If α is a number then α is an expression.
  - If α is an expression then also $(-\alpha)$ is an expression.
  - If α and β are expressions then also $(\alpha+\beta)$ is an expression.
  - If α and β are expressions then also $(\alpha-\beta)$ is an expression.
  - If α and β are expressions then also $(\alpha*\beta)$ is an expression.
  - If α and β are expressions then also (α/β) is an expression.

Context-Free Grammars

Example (cont.): The same information that was described by the previous inductive definition can be represented by a **context-free grammar**:

New auxiliary symbols, called **nonterminals**, are introduced:

- D — stands for an arbitrary digit
- C — stands for an arbitrary number
- E — stands for an arbitrary expression

$$
\begin{array}{llll}
D \rightarrow 0 & D \rightarrow 5 & & E \rightarrow C \\
D \rightarrow 1 & D \rightarrow 6 & & E \rightarrow (\text{-}E) \\
D \rightarrow 2 & D \rightarrow 7 & C \rightarrow D & E \rightarrow (E\text{+}E) \\
D \rightarrow 3 & D \rightarrow 8 & C \rightarrow DC & E \rightarrow (E\text{-}E) \\
D \rightarrow 4 & D \rightarrow 9 & & E \rightarrow (E*E) \\
 & & & E \rightarrow (E/E)
\end{array}
$$

Context-Free Grammars

Example (cont.): Written in a more succinct way:

$$
\begin{aligned}
D &\rightarrow \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \texttt{4} \mid \texttt{5} \mid \texttt{6} \mid \texttt{7} \mid \texttt{8} \mid \texttt{9} \\
C &\rightarrow D \mid DC \\
E &\rightarrow C \mid \texttt{(-}E\texttt{)} \mid \texttt{(}E\texttt{+}E\texttt{)} \mid \texttt{(}E\texttt{-}E\texttt{)} \mid \texttt{(}E\texttt{*}E\texttt{)} \mid \texttt{(}E\texttt{/}E\texttt{)}
\end{aligned}
$$

Context-Free Grammars

Example: A language where words are (possibly empty) sequences of expressions described in the previous example, where individual expressions are separated by commas (the alphabet must be extended with symbol “$,$”):

$$
\begin{aligned}
S &\rightarrow T \mid \varepsilon \\
T &\rightarrow E \mid E\,\texttt{,}\,T \\
D &\rightarrow \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \texttt{4} \mid \texttt{5} \mid \texttt{6} \mid \texttt{7} \mid \texttt{8} \mid \texttt{9} \\
C &\rightarrow D \mid DC \\
E &\rightarrow C \mid \texttt{(-}E\texttt{)} \mid \texttt{(}E\texttt{+}E\texttt{)} \mid \texttt{(}E\texttt{-}E\texttt{)} \mid \texttt{(}E\texttt{*}E\texttt{)} \mid \texttt{(}E\texttt{/}E\texttt{)}
\end{aligned}
$$

Context-Free Grammars

Example: Statements of some programming language (a fragment of a grammar):

$$
\begin{aligned}
S &\rightarrow E\texttt{;} \mid T \mid \texttt{if (}E\texttt{)}\ S \mid \texttt{if (}E\texttt{)}\ S\ \texttt{else}\ S \\
&\quad \mid \texttt{while (}E\texttt{)}\ S \mid \texttt{do}\ S\ \texttt{while (}E\texttt{);} \mid \texttt{for (}F\texttt{;}\,F\texttt{;}\,F\texttt{)}\ S \\
&\quad \mid \texttt{return}\ F\texttt{;} \\
T &\rightarrow \texttt{\{}\ U\ \texttt{\}} \\
U &\rightarrow \varepsilon \mid SU \\
F &\rightarrow \varepsilon \mid E \\
E &\rightarrow \quad \ldots
\end{aligned}
$$

Remark:

- S — statement
- T — block of statements
- U — sequence of statements
- E — expression
- F — optional expression that can be omitted

Context-Free Grammars

Formally, a **context-free grammar** is a tuple

$$G = (\Pi, \Sigma, S, P)$$

where:

- Π is a finite set of **nonterminal symbols** (**nonterminals**)
- Σ is a finite set of **terminal symbols** (**terminals**), where $\Pi \cap \Sigma = \emptyset$
- $S \in \Pi$ is an **initial nonterminal**
- $P \subseteq \Pi \times (\Pi \cup \Sigma)^*$ is a finite set of **rewrite rules**

Context-Free Grammars

Remarks:

- We will use uppercase letters A, B, C, ... to denote nonterminal symbols.
- We will use lowercase letters a, b, c, ... or digits 0, 1, 2, ... to denote terminal symbols.
- We will use lowercase Greek letters α, β, γ, ... do denote strings from $(\Pi \cup \Sigma)^*$.
- We will use the following notation for rules instead of (A, α)

$$A \rightarrow \alpha$$

A – left-hand side of the rule
α – right-hand side of the rule

Context-Free Grammars

Example: Grammar $G = (\Pi, \Sigma, S, P)$ where

- $\Pi = \{A, B, C\}$
- $\Sigma = \{a, b\}$
- $S = A$
- P contains rules

$$
\begin{aligned}
A &\longrightarrow aBBb \\
A &\longrightarrow AaA \\
B &\longrightarrow \varepsilon \\
B &\longrightarrow bCA \\
C &\longrightarrow AB \\
C &\longrightarrow a \\
C &\longrightarrow b
\end{aligned}
$$

Context-Free Grammars

Remark: If we have more rules with the same left-hand side, as for example

$$A \longrightarrow \alpha_1 \qquad\qquad A \longrightarrow \alpha_2 \qquad\qquad A \longrightarrow \alpha_3$$

we can write them in a more succinct way as

$$A \longrightarrow \alpha_1 \mid \alpha_2 \mid \alpha_3$$

For example, the rules of the grammar from the previous slide can be written as

$$\begin{aligned} A &\longrightarrow aBBb \mid AaA \\ B &\longrightarrow \varepsilon \mid bCA \\ C &\longrightarrow AB \mid a \mid b \end{aligned}$$

Context-Free Grammars

Grammars are used for generating words.

Example: $G = (\Pi, \Sigma, A, P)$ where $\Pi = \{A, B, C\}$, $\Sigma = \{a, b\}$, and P contains rules

$$\begin{aligned} A &\rightarrow aBBb \mid AaA \\ B &\rightarrow \varepsilon \mid bCA \\ C &\rightarrow AB \mid a \mid b \end{aligned}$$

For example, the word $abbabb$ can be in grammar G generated as follows:

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$\underline{A}$

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For example, the word $abbabb$ can be in grammar G generated as follows:

$\underline{A} \Rightarrow \underline{aBBb}$

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$A \Rightarrow aBBb$

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For example, the word $abbabb$ can be in grammar G generated as follows:

$A \Rightarrow a\underline{B}Bb$

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For example, the word $abbabb$ can be in grammar G generated as follows:

$A \Rightarrow a\underline{B}Bb \Rightarrow a\underline{bCA}Bb$

Context-Free Grammars

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Example: $G = (\Pi, \Sigma, A, P)$ where $\Pi = \{A, B, C\}$, $\Sigma = \{a, b\}$, and P contains rules

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For example, the word $abbabb$ can be in grammar G generated as follows:
$A \Rightarrow aBBb \Rightarrow abCABb$

Context-Free Grammars

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For example, the word $abbabb$ can be in grammar G generated as follows:

$A \Rightarrow aBBb \Rightarrow abC\underline{A}Bb$

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For example, the word $abbabb$ can be in grammar G generated as follows:

$A \Rightarrow aBBb \Rightarrow abC\underline{A}Bb \Rightarrow abC\underline{aBBb}Bb$

Context-Free Grammars

Grammars are used for generating words.

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For example, the word $abbabb$ can be in grammar G generated as follows:

$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb$

Context-Free Grammars

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Context-Free Grammars

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For example, the word $abbabb$ can be in grammar G generated as follows:

$$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb \Rightarrow ab\underline{C}aBbBb \Rightarrow ab\underline{b}aBbBb$$

Context-Free Grammars

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For example, the word $abbabb$ can be in grammar G generated as follows:

$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb \Rightarrow abCaBbBb \Rightarrow abbaBbBb \Rightarrow abba\underline{B}bb \Rightarrow abbabb$

Context-Free Grammars

Grammars are used for generating words.

Example: $G = (\Pi, \Sigma, A, P)$ where $\Pi = \{A, B, C\}$, $\Sigma = \{a, b\}$, and P contains rules

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Context-Free Grammars

On strings from $(\Pi \cup \Sigma)^*$ we define relation $\Rightarrow \subseteq (\Pi \cup \Sigma)^* \times (\Pi \cup \Sigma)^*$ such that

$$\alpha \Rightarrow \alpha'$$

iff $\alpha = \beta_1 A \beta_2$ and $\alpha' = \beta_1 \gamma \beta_2$ for some $\beta_1, \beta_2, \gamma \in (\Pi \cup \Sigma)^*$ and $A \in \Pi$ where $(A \to \gamma) \in P$.

Example: If $(B \to bCA) \in P$ then

$$aCBbA \Rightarrow aCbCAbA$$

Remark: Informally, $\alpha \Rightarrow \alpha'$ means that it is possible to derive α' from α by one step where an occurrence of some nonterminal A in α is replaced with the right-hand side of some rule $A \to \gamma$ with A on the left-hand side.

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Context-Free Grammars

A **derivation** of length n is a sequence $\beta_0, \beta_1, \beta_2, \cdots, \beta_n$, where $\beta_i \in (\Pi \cup \Sigma)^*$, and where $\beta_{i-1} \Rightarrow \beta_i$ for all $1 \leq i \leq n$, which can be written more succinctly as

$$\beta_0 \Rightarrow \beta_1 \Rightarrow \beta_2 \Rightarrow \ldots \Rightarrow \beta_{n-1} \Rightarrow \beta_n$$

The fact that for given $\alpha, \alpha' \in (\Pi \cup \Sigma)^*$ and $n \in \mathbb{N}$ there exists some derivation $\beta_0 \Rightarrow \beta_1 \Rightarrow \beta_2 \Rightarrow \ldots \Rightarrow \beta_{n-1} \Rightarrow \beta_n$, where $\alpha = \beta_0$ and $\alpha' = \beta_n$, is denoted

$$\alpha \Rightarrow^n \alpha'$$

The fact that $\alpha \Rightarrow^n \alpha'$ for some $n \geq 0$, is denoted

$$\alpha \Rightarrow^* \alpha'$$

Remark: Relation $\Rightarrow^*$ is the reflexive and transitive closure of relation $\Rightarrow$ (i.e., the smallest reflexive and transitive relation containing relation $\Rightarrow$).

Context-Free Grammars

Sentential forms are those $\alpha \in (\Pi \cup \Sigma)^*$, for which

$$S \Rightarrow^* \alpha$$

where S is the initial nonterminal.

Context-Free Grammars

A **language** $L(G)$ generated by a grammar $G = (\Pi, \Sigma, S, P)$ is the set of all words over alphabet Σ that can be derived by some derivation from the initial nonterminal S using rules from P, i.e.,

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

Context-Free Grammars

Example: We want to construct a grammar generating the language

$$L = \{a^n b^n \mid n \geq 0\}$$

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Grammar $G = (\Pi, \Sigma, S, P)$ where $\Pi = \{S\}$, $\Sigma = \{a, b\}$, and P contains

$$S \longrightarrow aSb \mid \varepsilon$$

$$\begin{array}{l}
S \Rightarrow \varepsilon \\
S \Rightarrow aSb \Rightarrow ab \\
S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb \\
S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb \\
S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaaaSbbbb \Rightarrow aaaabbbb \\
\qquad \ldots
\end{array}$$

Context-Free Grammars

Example: We want to construct a grammar generating the language consisting of all palindroms over the alphabet $\{a, b\}$, i.e.,

$$L = \{w \in \{a, b\}^* \mid w = w^R\}$$

Remark: w^R denotes the **reverse** of a word w, i.e., the word w written backwards.

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Solution:

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$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaaba$$

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Example: We want to construct a grammar generating the language L consisting of all correctly parenthesised sequences of symbols '(' and ')'.

For example $(()())(()) \in L$ but $)()) \notin L$.

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$$S \rightarrow \varepsilon \mid (S) \mid SS$$

Context-Free Grammars

Example: We want to construct a grammar generating the language L consisting of all correctly parenthesised sequences of symbols '$($' and '$)$'.

For example $(()())(()) \in L$ but $)()) \notin L$.

Solution:

$$S \rightarrow \varepsilon \mid (S) \mid SS$$

$S \Rightarrow SS \Rightarrow (S)S \Rightarrow (S)(S) \Rightarrow (SS)(S) \Rightarrow ((S)S)(S) \Rightarrow$
$(()S)(S) \Rightarrow (()(S))(S) \Rightarrow (()())(S) \Rightarrow (()())((S)) \Rightarrow$
$(()())(())$

Context-Free Grammars

Example: We want to construct a grammar generating the language L consisting of all correctly constructed arithmetic experessions where operands are always of the form 'a' and where symbols $+$ and $*$ can be used as operators.

For example $(a + a) * a + (a * a) \in L$.

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Solution:

$$E \rightarrow a \mid E + E \mid E * E \mid (E)$$

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For example $(a+a)*a+(a*a) \in L$.

Solution:

$$E \longrightarrow a \mid E+E \mid E*E \mid (E)$$

$E \Rightarrow E+E \Rightarrow E*E+E \Rightarrow (E)*E+E \Rightarrow (E+E)*E+E \Rightarrow$
$(a+E)*E+E \Rightarrow (a+a)*E+E \Rightarrow (a+a)*a+E \Rightarrow (a+a)*a+(E) \Rightarrow$
$(a+a)*a+(E*E) \Rightarrow (a+a)*a+(a*E) \Rightarrow (a+a)*a+(a*a)$

Derivation Tree

$$
\begin{aligned}
A &\rightarrow aBBb \mid AaA \\
B &\rightarrow \varepsilon \mid bCA \\
C &\rightarrow AB \mid a \mid b
\end{aligned}
$$

Derivation Tree

A

$$\begin{aligned} A &\rightarrow aBBb \mid AaA \\ B &\rightarrow \varepsilon \mid bCA \\ C &\rightarrow AB \mid a \mid b \end{aligned}$$

A

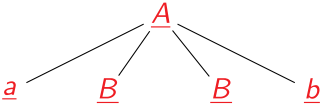
Derivation Tree

$\underline{A}$

$$\begin{aligned} \underline{A} &\rightarrow aBBb \mid AaA \\ B &\rightarrow \varepsilon \mid bCA \\ C &\rightarrow AB \mid a \mid b \end{aligned}$$

$\underline{A}$

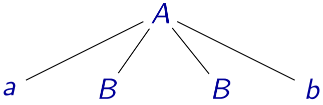
Derivation Tree



$$\begin{aligned}
\underline{A} &\rightarrow \underline{aBBb} \mid AaA \\
B &\rightarrow \varepsilon \mid bCA \\
C &\rightarrow AB \mid a \mid b
\end{aligned}$$

$\underline{A} \Rightarrow \underline{aBBb}$

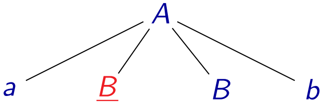
Derivation Tree



$$
\begin{aligned}
A &\longrightarrow aBBb \mid AaA\\
B &\longrightarrow \varepsilon \mid bCA\\
C &\longrightarrow AB \mid a \mid b
\end{aligned}
$$

$A \Rightarrow aBBb$

Derivation Tree



$$
\begin{aligned}
A &\longrightarrow aBBb \mid AaA \\
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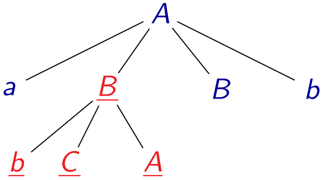
\end{aligned}
$$

$A \Rightarrow a\underline{B}Bb$

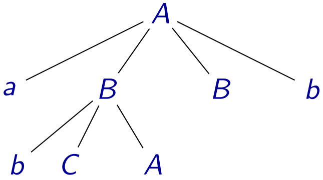
Derivation Tree



$$\begin{aligned} A &\longrightarrow aBBb \mid AaA \\ \underline{B} &\longrightarrow \varepsilon \mid \underline{bCA} \\ C &\longrightarrow AB \mid a \mid b \end{aligned}$$

$$A \Rightarrow a\underline{B}Bb \Rightarrow a\underline{bCA}Bb$$

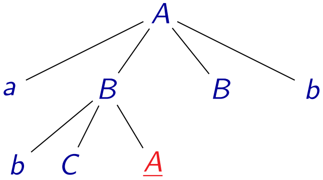
Derivation Tree



$A \rightarrow aBBb \mid AaA$
$B \rightarrow \varepsilon \mid bCA$
$C \rightarrow AB \mid a \mid b$

$A \Rightarrow aBBb \Rightarrow abCABb$

Derivation Tree



$\underline{A} \longrightarrow aBBb \mid AaA$
$B \longrightarrow \varepsilon \mid bCA$
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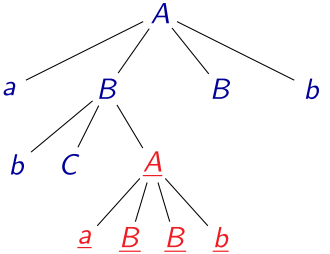
$A \Rightarrow aBBb \Rightarrow abC\underline{A}Bb$

Derivation Tree



$\underline{A} \longrightarrow \underline{aBBb} \mid AaA$
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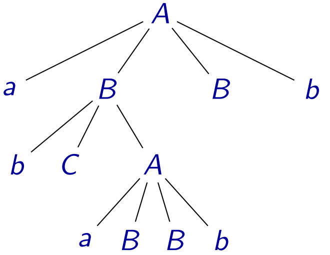
$$A \Rightarrow aBBb \Rightarrow abC\underline{A}Bb \Rightarrow abC\underline{aBBb}Bb$$

Derivation Tree

$$A \longrightarrow aBBb \mid AaA$$
$$B \longrightarrow \varepsilon \mid bCA$$
$$C \longrightarrow AB \mid a \mid b$$



$$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb$$

Derivation Tree

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\begin{aligned}
A &\longrightarrow aBBb \mid AaA \\
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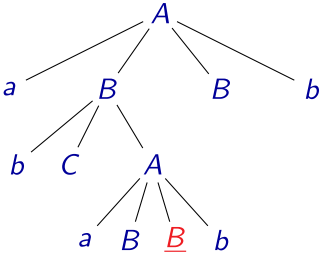
C &\longrightarrow AB \mid a \mid b
\end{aligned}
$$



$$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaB\underline{B}bBb$$

Derivation Tree

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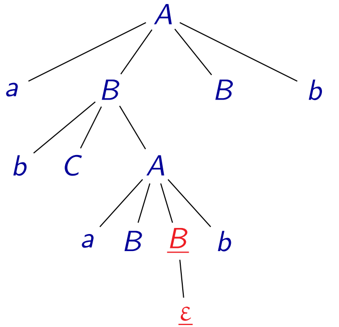

$$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaB\underline{B}bBb \Rightarrow abCaBbBb$$

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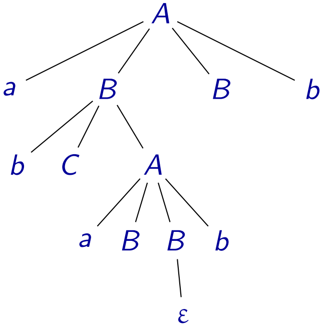

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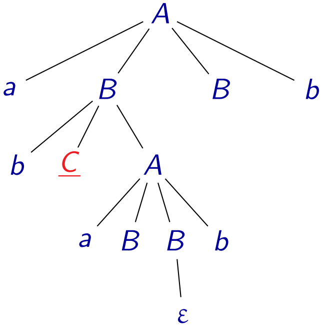

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Derivation Tree

$A \longrightarrow aBBb \mid AaA$
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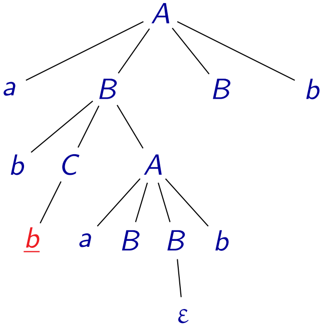
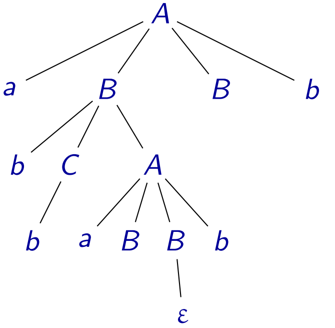
$\underline{C} \longrightarrow AB \mid a \mid \underline{b}$



$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb \Rightarrow ab\underline{C}aBbBb \Rightarrow ab\underline{b}aBbBb$

Derivation Tree

$$\begin{aligned} A &\rightarrow aBBb \mid AaA \\ B &\rightarrow \varepsilon \mid bCA \\ C &\rightarrow AB \mid a \mid b \end{aligned}$$



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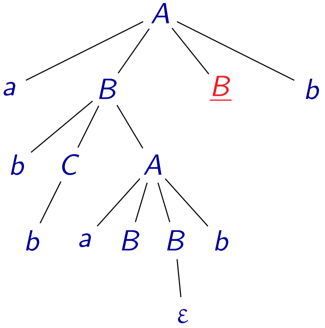

$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb \Rightarrow abCaBbBb \Rightarrow abbaBb\underline{B}b$

Derivation Tree

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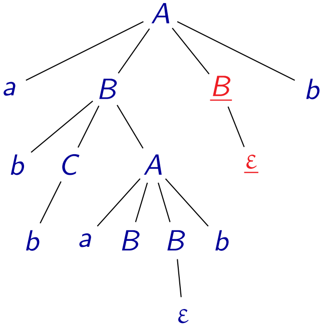
$$



$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb \Rightarrow abCaBbBb \Rightarrow abbaBb\underline{B}b \Rightarrow abbaBbb$

Derivation Tree

$A \longrightarrow aBBb \mid AaA$
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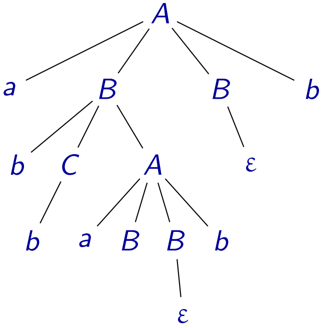

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Derivation Tree

$$A \longrightarrow aBBb \mid AaA$$
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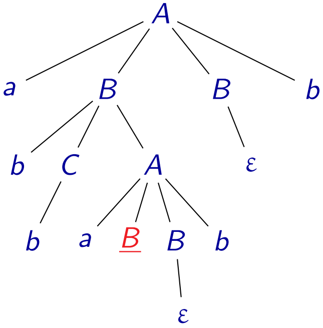
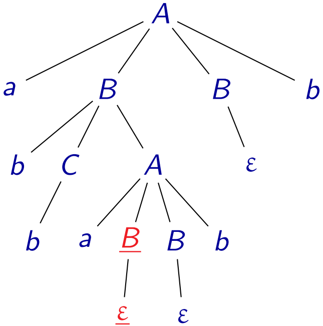


$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb \Rightarrow abCaBbBb \Rightarrow abbaBbBb \Rightarrow abba\underline{B}bb$

Derivation Tree

$$\begin{aligned} A &\longrightarrow aBBb \mid AaA \\ \underline{B} &\longrightarrow \underline{\varepsilon} \mid bCA \\ C &\longrightarrow AB \mid a \mid b \end{aligned}$$



$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb \Rightarrow abCaBbBb \Rightarrow abbaBbBb \Rightarrow abba\underline{B}bb \Rightarrow abbabb$

Derivation Tree

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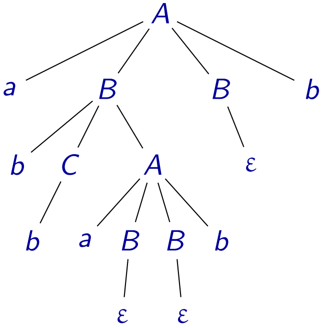


$A \Rightarrow aBBb \Rightarrow abCABb \Rightarrow abCaBBbBb \Rightarrow abCaBbBb \Rightarrow abbaBbBb \Rightarrow abbaBbb \Rightarrow abbabb$

Derivation Tree

For each derivation there is some **derivation tree**:

- Nodes of the tree are labelled with terminals and nonterminals.
- The root of the tree is labelled with the initial nonterminal.
- The leafs of the tree are labelled with terminals or with symbols ε.
- The remaining nodes of the tree are labelled with nonterminals.
- If a node is labelled with some nonterminal A then its children are labelled with the symbols from the right-hand side of some rewriting rule $A \rightarrow \alpha$.

Left and Right Derivation

$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

A **left derivation** is a derivation where in every step we always replace the leftmost nonterminal.

$$\underline{E} \Rightarrow \underline{E} + E \Rightarrow \underline{E} * E + E \Rightarrow a * \underline{E} + E \Rightarrow a * a + \underline{E} \Rightarrow a * a + a$$

A **right derivation** is a derivation where in every step we always replace the rightmost nonterminal.

$$\underline{E} \Rightarrow E + \underline{E} \Rightarrow \underline{E} + a \Rightarrow E * \underline{E} + a \Rightarrow \underline{E} * a + a \Rightarrow a * a + a$$

A derivation need not be left or right:

$$\underline{E} \Rightarrow \underline{E} + E \Rightarrow E * \underline{E} + E \Rightarrow E * a + \underline{E} \Rightarrow \underline{E} * a + a \Rightarrow a * a + a$$

Left and Right Derivation

- There can be several different derivations corresponding to one derivation tree.
- For every derivation tree, there is exactly one left and exactly one right derivation corresponding to the tree.

Equivalence of Grammars

Grammars G_1 and G_2 are **equivalent** if they generate the same language, i.e., if $L(G_1) = L(G_2)$.

Remark: The problem of equivalence of context-free grammars is algorithmically undecidable. It can be shown that it is not possible to construct an algorithm that would decide for any pair of context-free grammars if they are equivalent or not.

Even the problem to decide if a grammar generates the language Σ^* is algorithmically undecidable.

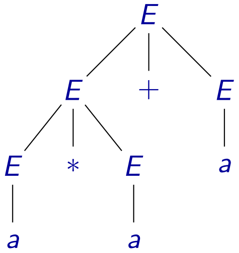
Ambiguous Grammars

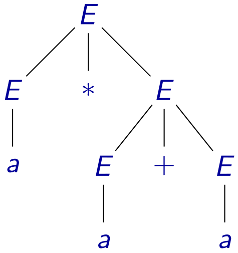
A grammar G is **ambiguous** if there is a word $w \in L(G)$ that has two different derivation trees, resp. two different left or two different right derivations.

Example:

$E \Rightarrow E + E \Rightarrow E * E + E \Rightarrow a * E + E \Rightarrow a * a + E \Rightarrow a * a + a$

$E \Rightarrow E * E \Rightarrow E * E + E \Rightarrow a * E + E \Rightarrow a * a + E \Rightarrow a * a + a$





Ambiguous Grammars

Sometimes it is possible to replace an ambiguous grammar with a grammar generating the same language but which is not ambiguous.

Example: A grammar

$$E \longrightarrow E + E \mid E * E \mid (E) \mid a$$

can be replaced with the equivalent grammar

$$\begin{aligned} E &\longrightarrow T \mid T + E \\ T &\longrightarrow F \mid F * T \\ F &\longrightarrow a \mid (E) \end{aligned}$$

Remark: If there is no unambiguous grammar equivalent to a given ambiguous grammar, we say it is **inherently ambiguous**.

Context-Free Languages

Definition

A language L is **context-free** if there exists some context-free grammar G such that $L = L(G)$.

The class of context-free languages is closed with respect to:

- concatenation
- union
- iteration

The class of context-free languages is not closed with respect to:

- complement
- intersection

Context-Free Languages

We have two grammars $G_1 = (\Pi_1, \Sigma, S_1, P_1)$ and $G_2 = (\Pi_2, \Sigma, S_2, P_2)$, and can assume that $\Pi_1 \cap \Pi_2 = \emptyset$ and $S \notin \Pi_1 \cup \Pi_2$.

- Grammar G such that $L(G) = L(G_1)L(G_2)$:

$$G = (\Pi_1 \cup \Pi_2 \cup \{S\},\ \Sigma,\ S,\ P_1 \cup P_2 \cup \{S \rightarrow S_1 S_2\})$$

- Grammar G such that $L(G) = L(G_1) \cup L(G_2)$:

$$G = (\Pi_1 \cup \Pi_2 \cup \{S\},\ \Sigma,\ S,\ P_1 \cup P_2 \cup \{S \rightarrow S_1, S \rightarrow S_2\})$$

- Grammar G such that $L(G) = L(G_1)^*$:

$$G = (\Pi_1 \cup \{S\},\ \Sigma,\ S,\ P_1 \cup \{S \rightarrow \varepsilon, S \rightarrow S_1 S\})$$

Algorithms

Algorithms

Example: An algorithm described by **pseudocode**:

Algorithm 1: An algorithm for finding the maximal element in an array

```
FIND-MAX ($A, n$):
begin
    $k := 0$
    for $i := 1$ to $n - 1$ do
        if $A[i] > A[k]$ then
            $k := i$
        end
    end
    return $A[k]$
end
```

Algorithms

Algorithm

- processes an **input**
- generates an **output**

From the point of view of an analysis how a given algorithm works, it usually makes only a little difference if the algorithm:

- reads input data from some input device (e.g., from a file, from a keyboard, etc.)
- writes data to some output device (e.g., to a file, on a screen, etc.)

or

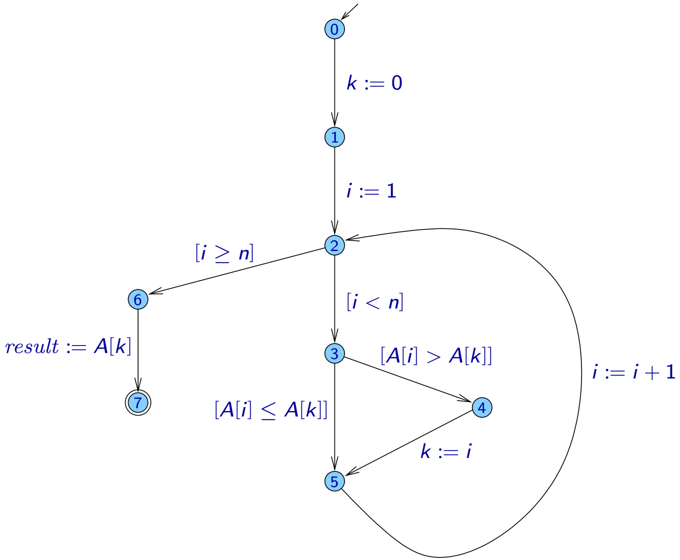
- reads input data from a memory (e.g., they are given to it as parameters)
- writes data somewhere to memory (e.g., it returns them as a return value)

Control Flow

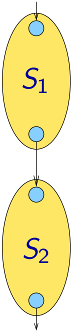
Intructions can be roughly devided into two groups:

- instructions working directly with data:
  - assignment
  - evaluation of values of expressions in conditions
  - reading input, writing output
  - ...
- instruction affecting the **control flow** — they determine, which instructions will be executed, in what order, etc.:
  - branching (if, switch, ...)
  - cycles (while, do .. while, for, ...)
  - organisation of intructions into blocks
  - returns from subprograms (return, ...)
  - ...

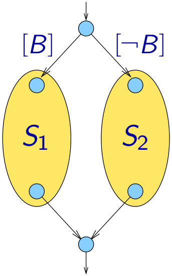
Control Flow Graph



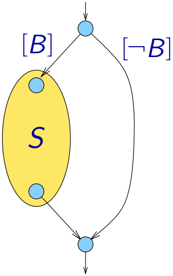
Some Basic Constructions of Structured Programming



$S_1; S_2$

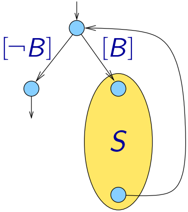


if B **then** S_1 **else** S_2

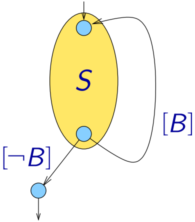


if B **then** S

Some Basic Constructions of Structured Programming

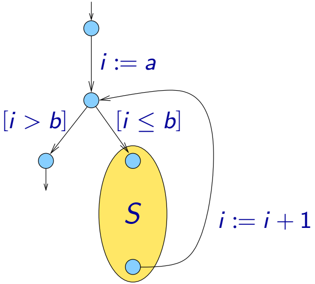


while B **do** S



do S **while** B

Some Basic Constructions of Structured Programming



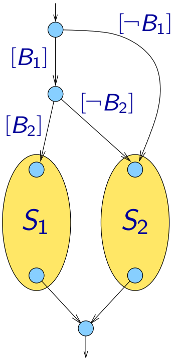
```
$i := a$
while $i \leq b$ do
    $S$
    $i := i + 1$
end
```

for $i := a$ **to** b **do** S

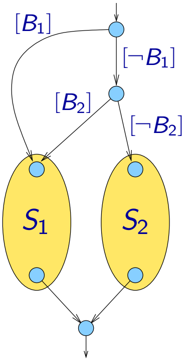
Some Basic Constructions of Structured Programming

Short-circuit evaluation of compound conditions, e.g.:

while $i < n$ **and** $A[i] > x$ **do** . . .



if B_1 **and** B_2 **then** S_1 **else** S_2



if B_1 **or** B_2 **then** S_1 **else** S_2

Control-flow Realized by GOTO

- **goto** ℓ — **unconditional jump**
- **if** B **then goto** ℓ — **conditional jump**

Example:

```
0:  $k := 0$
1:  $i := 1$
2:  goto 6
3:  if $A[i] \leq A[k]$ then goto 5
4:  $k := i$
5:  $i := i + 1$
6:  if $i < n$ then goto 3
7:  return $A[k]$
```

Control-flow Realized by GOTO

- **goto** ℓ — **unconditional jump**
- **if** B **then goto** ℓ — **conditional jump**

Example:

```
start:  $k := 0$
        $i := 1$
        goto $L3$
   L1:  if $A[i] \leq A[k]$ then goto $L2$
        $k := i$
   L2:  $i := i + 1$
   L3:  if $i < n$ then goto $L1$
        return $A[k]$
```

Evaluation of Complicated Expressions

Evaluation of a complicated expression such as

$$A[i+s] := (B[3*j+1]+x)*y+8$$

can be replaced by a sequence of simpler intructions on the lower level, such as

$$\begin{aligned}
&t_1 := i+s\\
&t_2 := 3*j\\
&t_2 := t_2+1\\
&t_3 := B[t_2]\\
&t_3 := t_3+x\\
&t_3 := t_3*y\\
&t_3 := t_3+8\\
&A[t_1] := t_3
\end{aligned}$$

Computation of an Algorithm

An algorithm is executed by a machine — it can be for example:

- real computer — executes intructions of a machine code
- virtual machine — executes instructions of a bytecode
- some idealized mathematical model of a computer
- ...

The machine can be:

- specialized — executes only one algorithm
- universal — can execute arbitrary algorithm, given in a form of **program**

The machine performs **steps**.

The algorithm processes a particular **input** during its computation.

Computation of an Algorithm

During a computation, the machine must remember:

- the current instruction
- the content of its working memory

It depends on the type of the machine:

- what is the type of data, with which the machine works
- how this data are organized in its memory

Depending on the type of the algorithm and the type of analysis, which we want to do, we can decide if it makes sense to include in memory also the places

- from which the input data are read
- where the output data are written

Computation of an Algorithm

Configuration — the description of the global state of the machine in some particular step during a computation

Example: A configuration of the form

$$(q, mem)$$

where

- q — the current control state
- mem — the current content of memory of the machine — the values assigned currently to variables.

An example of a content of memory mem:

$$\langle A\colon\ [3,8,1,3,6],\quad n\colon\ 5,\quad i\colon\ 1,\quad k\colon\ 0,\quad result\colon\ ?\rangle$$

Computation of an Algorithm

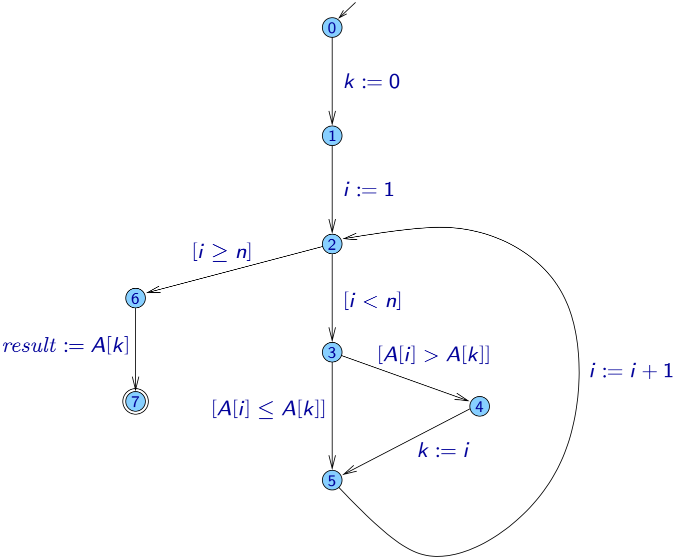
An example of a configuration:

$$(2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle)$$

A **computation** of a machine $\mathcal{M}$ executing an algorithm Alg, where it processes an input w, in a sequence of configurations.

- It starts in an **initial configuration**.
- In every step, it goes from one configuration to another.
- The computation ends in a **final configuration**.

Computation of an Algorithm



Computation of an Algorithm

Example: A computation, where algorithm Find-Max processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

α_0: $(0,\ \langle A\colon [3, 8, 1, 3, 6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon ?,\ \ result\colon ?\rangle)$

Computation of an Algorithm

Example: A computation, where algorithm FIND-MAX processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

α_0: $(0,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon ?,\ \ result\colon ?\rangle)$
α_1: $(1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ result\colon ?\rangle)$

Computation of an Algorithm

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α_2: $(2, \ \langle A\colon [3,8,1,3,6], \ \ n\colon 5, \ \ i\colon 1, \ \ k\colon 0, \ \ result\colon ?\rangle)$

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α_3: $(3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle)$

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α_4: $(4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ result\colon ?\rangle)$

Computation of an Algorithm

Example: A computation, where algorithm Find-Max processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

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α_1: $(1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle)$

α_2: $(2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle)$

α_3: $(3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle)$

α_4: $(4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle)$

α_5: $(5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle)$

Computation of an Algorithm

Example: A computation, where algorithm FIND-MAX processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

α_0: $(0,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } ?,\ \ k\text{: } ?,\ \ \mathit{result}\text{: } ?\rangle)$

α_1: $(1,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } ?,\ \ k\text{: } 0,\ \ \mathit{result}\text{: } ?\rangle)$

α_2: $(2,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 1,\ \ k\text{: } 0,\ \ \mathit{result}\text{: } ?\rangle)$

α_3: $(3,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 1,\ \ k\text{: } 0,\ \ \mathit{result}\text{: } ?\rangle)$

α_4: $(4,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 1,\ \ k\text{: } 0,\ \ \mathit{result}\text{: } ?\rangle)$

α_5: $(5,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 1,\ \ k\text{: } 1,\ \ \mathit{result}\text{: } ?\rangle)$

α_6: $(2,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 2,\ \ k\text{: } 1,\ \ \mathit{result}\text{: } ?\rangle)$

Computation of an Algorithm

Example: A computation, where algorithm Find-Max processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$\begin{array}{l}
\alpha_0\text{: } (0,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } ?,\ \ k\text{: } ?,\ \ \mathit{result}\text{: } ?\rangle) \\
\alpha_1\text{: } (1,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } ?,\ \ k\text{: } 0,\ \ \mathit{result}\text{: } ?\rangle) \\
\alpha_2\text{: } (2,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 1,\ \ k\text{: } 0,\ \ \mathit{result}\text{: } ?\rangle) \\
\alpha_3\text{: } (3,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 1,\ \ k\text{: } 0,\ \ \mathit{result}\text{: } ?\rangle) \\
\alpha_4\text{: } (4,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 1,\ \ k\text{: } 0,\ \ \mathit{result}\text{: } ?\rangle) \\
\alpha_5\text{: } (5,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 1,\ \ k\text{: } 1,\ \ \mathit{result}\text{: } ?\rangle) \\
\alpha_6\text{: } (2,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 2,\ \ k\text{: } 1,\ \ \mathit{result}\text{: } ?\rangle) \\
\alpha_7\text{: } (3,\ \langle A\text{: } [3,8,1,3,6],\ \ n\text{: } 5,\ \ i\text{: } 2,\ \ k\text{: } 1,\ \ \mathit{result}\text{: } ?\rangle)
\end{array}$$

Computation of an Algorithm

Example: A computation, where algorithm FIND-MAX processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$
\begin{array}{llllllll}
\alpha_0\colon & (0, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon ?, & k\colon ?, & \mathit{result}\colon ?\rangle) \\
\alpha_1\colon & (1, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon ?, & k\colon 0, & \mathit{result}\colon ?\rangle) \\
\alpha_2\colon & (2, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 1, & k\colon 0, & \mathit{result}\colon ?\rangle) \\
\alpha_3\colon & (3, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 1, & k\colon 0, & \mathit{result}\colon ?\rangle) \\
\alpha_4\colon & (4, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 1, & k\colon 0, & \mathit{result}\colon ?\rangle) \\
\alpha_5\colon & (5, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 1, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_6\colon & (2, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 2, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_7\colon & (3, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 2, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_8\colon & (5, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 2, & k\colon 1, & \mathit{result}\colon ?\rangle)
\end{array}
$$

Computation of an Algorithm

Example: A computation, where algorithm FIND-MAX processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$
\begin{array}{l}
\alpha_0\colon\ (0,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon ?,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_1\colon\ (1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_2\colon\ (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_3\colon\ (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_4\colon\ (4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_5\colon\ (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_6\colon\ (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_7\colon\ (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_8\colon\ (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_9\colon\ (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle)
\end{array}
$$

Computation of an Algorithm

Example: A computation, where algorithm $\textsc{Find-Max}$ processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$
\begin{array}{rl}
\alpha_0: & (0,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon ?,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_1: & (1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_2: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_3: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_4: & (4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_5: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_6: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_7: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_8: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_9: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{10}: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle)
\end{array}
$$

Computation of an Algorithm

Example: A computation, where algorithm FIND-MAX processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$
\begin{array}{rl}
\alpha_0\colon & (0,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon ?,\ \ result\colon ?\rangle) \\
\alpha_1\colon & (1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ result\colon ?\rangle) \\
\alpha_2\colon & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ result\colon ?\rangle) \\
\alpha_3\colon & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ result\colon ?\rangle) \\
\alpha_4\colon & (4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ result\colon ?\rangle) \\
\alpha_5\colon & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 1,\ \ result\colon ?\rangle) \\
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\alpha_7\colon & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ result\colon ?\rangle) \\
\alpha_8\colon & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ result\colon ?\rangle) \\
\alpha_9\colon & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ result\colon ?\rangle) \\
\alpha_{10}\colon & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ result\colon ?\rangle) \\
\alpha_{11}\colon & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ result\colon ?\rangle)
\end{array}
$$

Computation of an Algorithm

Example: A computation, where algorithm $\textsc{Find-Max}$ processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$
\begin{array}{rl}
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\alpha_1: & (1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_2: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_3: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_4: & (4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_5: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_6: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_7: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_8: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_9: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{10}: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{11}: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{12}: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle)
\end{array}
$$

Computation of an Algorithm

Example: A computation, where algorithm FIND-MAX processes an input where $A = [3,8,1,3,6]$ and $n = 5$.

$$
\begin{array}{rl}
\alpha_0: & (0,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon ?,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_1: & (1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_2: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_3: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_4: & (4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_5: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_6: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_7: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_8: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_9: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{10}: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{11}: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{12}: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{13}: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle)
\end{array}
$$

Computation of an Algorithm

Example: A computation, where algorithm FIND-MAX processes an input where $A = [3,8,1,3,6]$ and $n = 5$.

$$
\begin{array}{rl}
\alpha_0\colon & (0,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon ?,\ k\colon ?,\ \mathit{result}\colon ?\rangle) \\
\alpha_1\colon & (1,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon ?,\ k\colon 0,\ \mathit{result}\colon ?\rangle) \\
\alpha_2\colon & (2,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 1,\ k\colon 0,\ \mathit{result}\colon ?\rangle) \\
\alpha_3\colon & (3,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 1,\ k\colon 0,\ \mathit{result}\colon ?\rangle) \\
\alpha_4\colon & (4,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 1,\ k\colon 0,\ \mathit{result}\colon ?\rangle) \\
\alpha_5\colon & (5,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 1,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_6\colon & (2,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 2,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_7\colon & (3,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 2,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_8\colon & (5,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 2,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_9\colon & (2,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 3,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_{10}\colon & (3,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 3,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_{11}\colon & (5,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 3,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_{12}\colon & (2,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 4,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_{13}\colon & (3,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 4,\ k\colon 1,\ \mathit{result}\colon ?\rangle) \\
\alpha_{14}\colon & (5,\ \langle A\colon [3,8,1,3,6],\ n\colon 5,\ i\colon 4,\ k\colon 1,\ \mathit{result}\colon ?\rangle)
\end{array}
$$

Computation of an Algorithm

Example: A computation, where algorithm $\textsc{Find-Max}$ processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$
\begin{array}{rlllllll}
\alpha_0: & (0, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon ?, & k\colon ?, & \mathit{result}\colon ?\rangle) \\
\alpha_1: & (1, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon ?, & k\colon 0, & \mathit{result}\colon ?\rangle) \\
\alpha_2: & (2, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 1, & k\colon 0, & \mathit{result}\colon ?\rangle) \\
\alpha_3: & (3, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 1, & k\colon 0, & \mathit{result}\colon ?\rangle) \\
\alpha_4: & (4, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 1, & k\colon 0, & \mathit{result}\colon ?\rangle) \\
\alpha_5: & (5, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 1, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_6: & (2, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 2, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_7: & (3, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 2, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_8: & (5, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 2, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_9: & (2, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 3, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_{10}: & (3, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 3, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_{11}: & (5, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 3, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_{12}: & (2, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 4, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_{13}: & (3, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 4, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_{14}: & (5, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 4, & k\colon 1, & \mathit{result}\colon ?\rangle) \\
\alpha_{15}: & (2, & \langle A\colon [3,8,1,3,6], & n\colon 5, & i\colon 5, & k\colon 1, & \mathit{result}\colon ?\rangle)
\end{array}
$$

Computation of an Algorithm

Example: A computation, where algorithm Find-Max processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$\begin{array}{rl}
\alpha_0: & (0,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon ?,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_1: & (1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_2: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_3: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_4: & (4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_5: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_6: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_7: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_8: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_9: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{10}: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{11}: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{12}: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{13}: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{14}: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{15}: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 5,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{16}: & (6,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 5,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle)
\end{array}$$

Computation of an Algorithm

Example: A computation, where algorithm $\textsc{Find-Max}$ processes an input where $A = [3, 8, 1, 3, 6]$ and $n = 5$.

$$\begin{array}{rl}
\alpha_0: & (0,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon ?,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_1: & (1,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon ?,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_2: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_3: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_4: & (4,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 0,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_5: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 1,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_6: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_7: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_8: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 2,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_9: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{10}: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{11}: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 3,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{12}: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{13}: & (3,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{14}: & (5,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 4,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{15}: & (2,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 5,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{16}: & (6,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 5,\ \ k\colon 1,\ \ \mathit{result}\colon ?\rangle) \\
\alpha_{17}: & (7,\ \langle A\colon [3,8,1,3,6],\ \ n\colon 5,\ \ i\colon 5,\ \ k\colon 1,\ \ \mathit{result}\colon 8\rangle)
\end{array}$$

Computation of an Algorithm

By executing an instruction I, the machine goes from configuration α to configuration α':

$$\alpha \xrightarrow{I} \alpha'$$

A computation can be:

- **Finite**:

  $$\alpha_0 \xrightarrow{I_0} \alpha_1 \xrightarrow{I_1} \alpha_2 \xrightarrow{I_2} \alpha_3 \xrightarrow{I_3} \alpha_4 \xrightarrow{I_4} \quad \cdots \quad \xrightarrow{I_{t-2}} \alpha_{t-1} \xrightarrow{I_{t-1}} \alpha_t$$

  where α_t is a final configuration

- **Infinite**:

  $$\alpha_0 \xrightarrow{I_0} \alpha_1 \xrightarrow{I_1} \alpha_2 \xrightarrow{I_2} \alpha_3 \xrightarrow{I_3} \alpha_4 \xrightarrow{I_4} \quad \cdots$$

Computation of an Algorithm

A computation can be described in two different ways:

- as a sequence of configurations $\alpha_0, \alpha_1, \alpha_2, \ldots$
- as a sequence of executed instructions $I_0, I_1, I_2, \ldots$

Correctness of Algorithms

Algorithms are used for solving **problems**.

- **Problem** — a specification **what** should be computed by an algorithm:
  - Description of inputs
  - Description of outputs
  - How outputs are related to inputs
- **Algorithm** — a particular procedure that describes **how** to compute an output for each possible input

Correctness of Algorithms

Example: The problem of finding a maximal element in an array:

Input: An array A indexed from zero and a number n representing the number of elements in array A. It is assumed that $n \geq 1$.

Output: A value $result$ of a maximal element in the array A, i.e., the value $result$ such that:

- $A[j] \leq result$ for all $j \in \mathbb{N}$, where $0 \leq j < n$, and
- there exists $j \in \mathbb{N}$ such that $0 \leq j < n$ and $A[j] = result$.

An **instance** of a problem — concreate input data, e.g.,

$$A = [3, 8, 1, 3, 6], \ n = 5.$$

The output for this instance is value 8.

Correctness of Algorithms

Definition

An algorithm Alg **solves** a given problem P, if for instance w of problem P, the following conditions are satisfied:

(a) The computation of algorithm Alg on input w halts after finite number of steps.

(b) Algorithm Alg generates a correct output for input w according to conditions in problem P.

An algorithm that solves problem P is a correct solution of this problem.

Correctness of Algorithms

Algorithm Alg is **not** a correct solution of problem P if there exists an input w such that in the computation on this input, one of the following incorrect behaviours occurs:

- some incorrect illegal operation is performed (an access to an element of an array with index out of bounds, division by zero, . . .),
- the generated output does not satisfy the conditions specified in problem P,
- the computation never halts.

Testing — running the algorithm with different inputs and checking whether the algorithm behaves correctly on these inputs.

Testing can be used to show the presence of bugs but not to show that algorithm behaves correctly for **all** inputs.

Correctness of Algorithms

Generally, it is reasonable to divide a proof of correctness of an algorithm into two parts:

- Showing that the algorithm never does anything “wrong” for any input:
  - no illegal operation is performed during a computation
  - if the program halts, the generated output will be “correct”
- Showing that for every input the algorithm halts after a finite number of steps.

Correctness of Algorithms

Invariant — a condition that must be always satisfied in a given position in a code of the algorithm (i.e., in all possible computations for all allowed inputs) whenever the algorithm goes through this position.

We say that a configuration α is **reachable** if there exists an input w such that α is one of configurations through which the algorithm goes in the computation on input w.

If an algorithm is represented by a control-flow graph, for a given **control state** q (i.e., a node of the graph) we can specify invariants that hold in every reachable configuration with control state q.

Invariants

Invariants can be written as formulas of predicate logic:

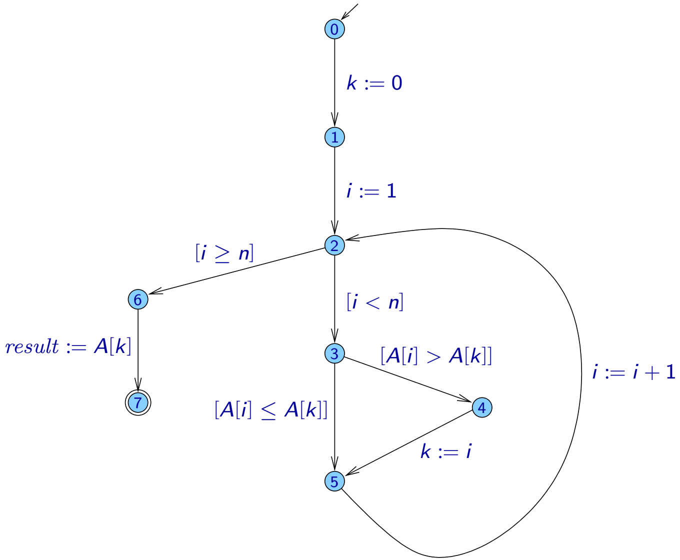
- **free** variables correspond to variables of the program
- a **valuation** is determined by values of program variables in a given configuration

Example: Formula

$$(1 \leq i) \land (i \leq n)$$

holds for example in a configuration where variable i has value 5 and variable n has value 14.

Invariants



Invariants

Examples of invariants:

- an invariant in a control state q is represented by a formula φ_q

Invariants for individual control states (so far only hypotheses):

- φ_0: $(n \geq 1)$
- φ_1: $(n \geq 1) \wedge (k = 0)$
- φ_2: $(n \geq 1) \wedge (1 \leq i \leq n) \wedge (0 \leq k < i)$
- φ_3: $(n \geq 1) \wedge (1 \leq i < n) \wedge (0 \leq k < i)$
- φ_4: $(n \geq 1) \wedge (1 \leq i < n) \wedge (0 \leq k < i)$
- φ_5: $(n \geq 1) \wedge (1 \leq i < n) \wedge (0 \leq k \leq i)$
- φ_6: $(n \geq 1) \wedge (i = n) \wedge (0 \leq k < n)$
- φ_7: $(n \geq 1) \wedge (i = n) \wedge (0 \leq k < n)$

Invariants

Examples of invariants:

- an invariant in a control state q is represented by a formula φ_q

Invariants for individual control states (so far only hypotheses):

- φ_0: $n \geq 1$
- φ_1: $n \geq 1, \ \ k = 0$
- φ_2: $n \geq 1, \ \ 1 \leq i \leq n, \ \ 0 \leq k < i$
- φ_3: $n \geq 1, \ \ 1 \leq i < n, \ \ 0 \leq k < i$
- φ_4: $n \geq 1, \ \ 1 \leq i < n, \ \ 0 \leq k < i$
- φ_5: $n \geq 1, \ \ 1 \leq i < n, \ \ 0 \leq k \leq i$
- φ_6: $n \geq 1, \ \ i = n, \ \ 0 \leq k < n$
- φ_7: $n \geq 1, \ \ i = n, \ \ 0 \leq k < n$

Invariants

Checking that the given invariants really hold:

- It is necessary to check for each instruction of the algorithm that under the assumption that a specified invariant holds before an execution of the instruction, the other specified invariant holds after the execution of the instruction.

Let us assume the algorithm is represented as a control-flow graph:

- edges correspond to instructions
- consider an edge from state q to state q' labelled with instruction I
- let us say that (so far non-verified) invariants for states q and q' are expressed by formulas φ and φ'
- for this edge we must check that for every configurations $\alpha = (q, mem)$ and $\alpha' = (q', mem')$ such that $\alpha \xrightarrow{I} \alpha'$, it holds that if
  - φ holds is configuration α,

  then
  - φ' holds in configuration α'

Invariants

Checking instructions, which are conditional tests:

- an edge labelled with a conditional test $[B]$

A content of memory is not modified.
It is sufficient to check that the following implication holds

$$(\varphi \wedge B) \rightarrow \varphi'$$

Remark: The given implication must hold for all possible values of variables.

Example: Let us assume that formulas contain only variables n, i, k, and that values of these variables are integers:

$$(\forall n \in \mathbb{Z})(\forall i \in \mathbb{Z})(\forall k \in \mathbb{Z})\,(\varphi \wedge B \rightarrow \varphi')$$

Invariants

Checking those instructions that assign values to variables (they modify a content of memory):

- an edge labelled with assignment $x := E$

φ'' — a formula obtained from formula φ' by renaming of all free occurrences of variable x to x'

It is necessary to check the validity of implication

$$(\varphi \wedge (x' = E)) \rightarrow \varphi''$$

Example: Assignment $k := 3 * k + i + 1$:

$$(\forall n \in \mathbb{Z})(\forall i \in \mathbb{Z})(\forall k \in \mathbb{Z})(\forall k' \in \mathbb{Z})\,(\varphi \wedge (k' = 3 * k + i + 1) \rightarrow \varphi'')$$

Invariants

Finishing the checking that the algorithm for finding maximal element in an array returns a correct result (under assumption that it halts):

- ψ_0: φ_0
- ψ_1: $\varphi_1 \wedge (\forall j \in \mathbb{N})(0 \leq j < 1 \rightarrow A[j] \leq A[k])$
- ψ_2: $\varphi_2 \wedge (\forall j \in \mathbb{N})(0 \leq j < i \rightarrow A[j] \leq A[k])$
- ψ_3: $\varphi_3 \wedge (\forall j \in \mathbb{N})(0 \leq j < i \rightarrow A[j] \leq A[k])$
- ψ_4: $\varphi_4 \wedge (\forall j \in \mathbb{N})(0 \leq j < i \rightarrow A[j] \leq A[k]) \wedge (A[i] > A[k])$
- ψ_5: $\varphi_5 \wedge (\forall j \in \mathbb{N})(0 \leq j \leq i \rightarrow A[j] \leq A[k])$
- ψ_6: $\varphi_6 \wedge (\forall j \in \mathbb{N})(0 \leq j < n \rightarrow A[j] \leq A[k])$
- ψ_7: $\varphi_7 \wedge (\mathit{result} = A[k]) \wedge (\forall j \in \mathbb{N})(0 \leq j < n \rightarrow A[j] \leq \mathit{result}) \wedge (\exists j \in \mathbb{N})(0 \leq j < n \wedge A[j] = \mathit{result})$

Invariants

Usually it is not necessary to specify invariants in all control states but only in some “important” states — in particular, in states where the algorithm enters or leaves loops:

It is necessary to verify:

- That the invariant holds before entering the loop.
- That if the invariant holds before an iteration of the loop then it holds also after the iteration.
- That the invariant holds when the loop is left.

Invariants

Example: In algorithm FIND-MAX, state 2 is such "important" state.

In state 2, the following holds:

- $n \geq 1$
- $1 \leq i \leq n$
- $0 \leq k < i$
- For each j such that $0 \leq j < i$ it holds that $A[j] \leq A[k]$.

Finiteness of a Computation

Two possibilities how an infinite computation can look:

- some configuration is repeated — then all following configurations are also repeated
- all configurations in a computation are different but a final configuration is never reached

Finiteness of a Computation

One of standard ways of proving that an algorithm halts for every input after a finite number of steps:

- to assign a value from a set W to every (reachable) configuration
- to define an order $\leq$ on set W such that there are no infinite (strictly) decreasing sequences of elements of W
- to show that the values assigned to configuration decrease with every execution of each instruction, i.e., if $\alpha \xrightarrow{I} \alpha'$ then

  $$f(\alpha) > f(\alpha')$$

  ($f(\alpha)$, $f(\alpha')$ are values from set W assigned to configurations α and α')

Finiteness of a Computation

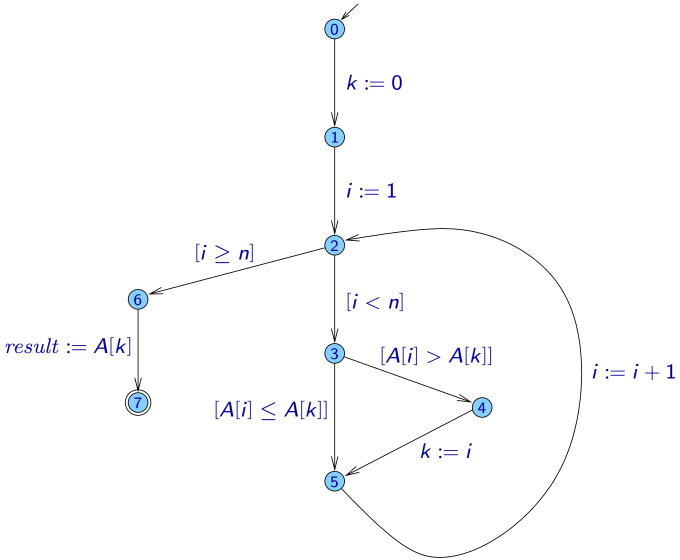
As a set W, we can use for example:

- The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ with ordering $\leq$.
- The set of vectors of natural numbers with lexicographic ordering, i.e., the ordering where vector $(a_1, a_2, \dots, a_m)$ is smaller than $(b_1, b_2, \dots, b_n)$, if
  - there exists i such that $1 \leq i \leq m$ and $i \leq n$, where $a_i < b_i$ and for all j such that $1 \leq j < i$ it holds that $a_j = b_j$, or
  - $m < n$ and for all j such that $1 \leq j \leq m$ is $a_j = b_j$.

For example, $(5, 1, 3, 6, 4) < (5, 1, 4, 1)$ and $(4, 1, 1) < (4, 1, 1, 3)$.

Remark: The number of elemets in vectors must be bounded by some constant.

Finiteness of a Computation



Finiteness of a Computation

Example: Vectors assigned to individual configurations:

- State 0: $f(\alpha) = (4)$
- State 1: $f(\alpha) = (3)$
- State 2: $f(\alpha) = (2, n - i, 3)$
- State 3: $f(\alpha) = (2, n - i, 2)$
- State 4: $f(\alpha) = (2, n - i, 1)$
- State 5: $f(\alpha) = (2, n - i, 0)$
- State 6: $f(\alpha) = (1)$
- State 7: $f(\alpha) = (0)$

Computational Complexity of Algorithms

Complexity of an Algorithm

- Computers work fast but not infinitely fast. Execution of each instruction takes some (very short) time.
- The same problem can be solved by several different algorithms. The time of a computation (determined mostly by the number of executed instructions) can be different for different algorithms.
- We can implement the algorithms and then measure the time of their computation. By this we find out how long the computation takes on particular data on which we test the algorithm.
- We would like to have a more precise idea how long the computation takes on all possible input data.

Complexity of an Algorithm

- A running time is affected by many factors, e.g.:
    - the algorithm that is used
    - the amount of input data
    - used hardware (e.g., the frequency at which a CPU is running can be important)
    - the used programming language — its implementation (compiler/interpreter)
    - ...

- If we need to solve problem for “small” input data, the running time is usually negligible.

- With increasing amount of input data (the size of input), the running time can grow, sometimes significantly.

Complexity of an Algorithm

- **Time complexity of an algorithm** — how the running time of the algorithm depends on the amount of input data
- **Space complexity of an algorithm** — how the amount of a memory used during a computation grows with respect to the size of input

Remark: The precise definitions will be given later.

Remark:

- There are also other types of computational complexity, which we will not discuss here (e.g., communication complexity).

Complexity of an Algorithm

To determine the precise running time or the precise amount of used memory just by an analysis of an algorithm can be extremely difficult.

Usually the analysis of complexity of an algorithm involves many simplifications:

- It is usually not analysed how the running time or the amount of used memory depends precisely on particular input data but how they depend on the **size of the input**.
- Functions expressing how the running time or the amount of used memory grows depending on the size of the input are not computed precisely — instead **estimations** of these functions are computed.
- Estimations of these functions are usually expressed using **asymptotic notation** — e.g., it can be said that the running time of MergeSort is $O(n \log n)$, and that the running time of BubbleSort is $O(n^2)$.

Size of Input

Size of the input — a value describing how “big” is an input instance

- In most cases, the size of an input is just one number — it is usually denoted n or N.
- Sometimes it is more appropriate to express the size of an input by pair (sometimes even with three, four, etc.) of parameters — in this case, they are ofted denoted n and m (or N and M).
- We can choose what should be considered as the size of an input.

Size of Input

Examples, what the size of an input can be:

- An input is a sequence of some values, an array of elements, etc. (e.g., in problems like sorting, searching in an array, finding the maximal element, etc.):

  n — the number of elements in this sequence or array

- An input is a string of characters (a word from some alphabet):

  n — the number of characters in this string

- An input consists of two strings, e.g., a (long) text that will be searched through, and a (shorter) searched pattern:

  n — the number of characters in the text
  m — the number of characters in the searched pattern

Size of Input

- An input is a set of strings:

  One possibility:

  n — the sum of lengths of all strings

  Other variant:

  n — the sum of lengths of all strings, m — the number of strings

- The input is a graph:

  n — the number of nodes, m — the number of edges

Size of Input

- The input is one number (e.g., in the primarity testing):

  One possibility:

  n — the number of bits of the number — e.g., the size of input 962261 is 20

  Other variant:

  n — the value of the number — the size of input 962261 is 962261

- The input is a sequence of numbers, and the running time is affected by the values of the numbers (e.g., in the problem where the goal is to compute the greatest common divisor of all numbers in a given sequence):

  n — the sum of numbers of bits of all numbers in the given sequence

Running Time

Let us say that we have:

- an algorithm Alg solving a problem P (resp. a particular implementation of algorithm Alg),
- a machine $\mathcal{M}$ executing the algorithm Alg,
- an input w from the set In, which is a set of all inputs of problem P

An example:

- a particular implementation of Quicksort in C++ solving the problem of sorting,
- a computer with some particular type of processor working on some particular frequency, with some particular amount of memory, operating system, etc.
- input: array $[6, 13, 1, 8, 4, 5, 8]$
(remark: a more realistic example would be an array with one million elements)

Running Time

$t(w)$ — the running time of the algorithm Alg on input w on machine $\mathcal{M}$

What units should be used for expressing time? (As we will see, this is not important when asymptotic notation is used.)

- **in seconds** — it depends on too many details of implementation, it is difficult to determine it in other way than by measurement

  (even on the same computer with the same data the running time can fluctuate)

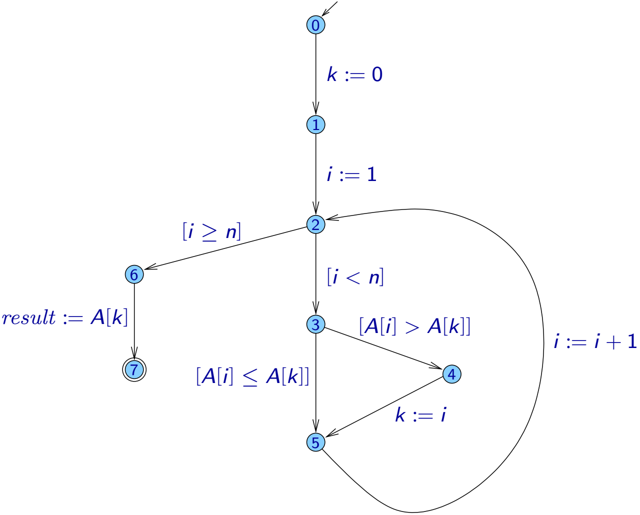
- **the number of steps** — it must be specified what is considered as one step, for example:
  - one statement of a high level programming language
  - one instruction of machine code or bytecode
  - one tick of a processor
  - one operation of some particular type — e.g., a comparison, an arithmetic operation, etc. (while all other operations are ignored)
  - . . .

Running Time

Let us say that an algorithm is represented by a control-flow graph:

- To every instruction (i.e., to every edge) we assign a value specifying how long it takes to perform this instruction once.
- The execution time of different instructions can be different.
- For simplicity we assume that an execution of the same instruction takes always the same time — the value assigned to an instruction is a number from the set $\mathbb{R}^+$ (the set of nonnegative real numbers).

Running Time



Instr.	time
$k := 0$	c_0
$i := 1$	c_1
$[i < n]$	c_2
$[i \geq n]$	c_3
$[A[i] \leq A[k]]$	c_4
$[A[i] > A[k]]$	c_5
$k := i$	c_6
$i := i + 1$	c_7
$result := A[k]$	c_8

Running Time

Example: The execution times of individual instructions could be for example:

Instr.	symbol	time
$k := 0$	c_0	4
$i := 1$	c_1	4
$[i < n]$	c_2	10
$[i \geq n]$	c_3	12
$[A[i] \leq A[k]]$	c_4	14
$[A[i] > A[k]]$	c_5	12
$k := i$	c_6	5
$i := i + 1$	c_7	6
$\mathit{result} := A[k]$	c_8	5

For a particular input w, e.g., for $w = ([3, 8, 4, 5, 2], 5)$, we could simulate the computation and determine the precise running time $t(w)$.

Time Complexity of an Algorithm

Let us say that:

- For a given algorithm Alg and machine $\mathcal{M}$, and for every input w from the set of all inputs In, the running time $t(w)$ is precisely defined.
- To each input w from set In, a number $size(w)$ describing the size of the input w is assigned .

  (Formally, it is a function $size : In \rightarrow \mathbb{N}$.)

Definition

The **time complexity of algorithm Alg in the worst case** is the function $T : \mathbb{N} \rightarrow \mathbb{R}^+$that assigns to each natural number n the maximal running time of the algorithm Alg on an input of size n.

So for each $n \in \mathbb{N}$ we have:

- For each input $w \in In$ such that $size(w) = n$ is $t(w) \leq T(n)$.
- There exists an input $w \in In$ such that $size(w) = n$ and $t(w) = T(n)$.

Time Complexity of an Algorithm

It is obvious from this definition that the time complexity of an algorithm is a function whose precise values depend not only on the given algorithm Alg but also on the following things:

- on a machine $\mathcal{M}$, on which the algorithm Alg runs,
- on the precise definition of the running time $t(w)$ of algorithm Alg on machine $\mathcal{M}$ with input $w \in In$,
- on the precise definition of the size of an input (i.e., on the definition of function $size$).

Time Complexity of an Algorithm

Sometimes, the time complexity in the **average case** is also analyzed:

- Some particular **probabilistic distribution** on the set of inputs must be assumed.
- Instead of the maximal running time on inputs of size n, the expected value of the running times is considered.
- Usually, the analysis of the avarage case is much more complicated than the analysis of the worst case.
- Often, these two functions are not very different but sometimes the difference is significant.

Remark: It usually makes little sense to analyze the time complexity in the best case.

Time Complexity of an Algorithm

An example of an analysis of the time complexity of algorithm FIND-MAX **without** the use of asymptotic notation:

- Such precise analysis is almost never done in practice — it is too tedious and complicated.
- This illustrates what things are ignored in an analysis where asymptotic notation is used and how much the analysis is simplified by this.
- We will compute with constants $c_0, c_1, \ldots, c_8$, which specify the execution time of individual instructions — we won't compute with concrete numbers.

Time Complexity of an Algorithm

The inputs are of the form (A, n), where A is an array and n is the number of elements in this array (where $n \geq 1$).

We take n as the size of input (A, n).

Consider now some particular input $w = (A, n)$ of size n:

- The running time $t(w)$ on input w can be expressed as

  $$t(w) = c_0 m_0 + c_1 m_1 + \cdots + c_8 m_8,$$

  where $m_0, m_1, \ldots, m_8$ are numbers specifying how many times is each instruction performed in the computation on input w.

Time Complexity of an Algorithm

Instr.	time	occurences	value of m_i
$k := 0$a	c_0	m_0	1
$i := 1$	c_1	m_1	1
$[i < n]$	c_2	m_2	$n - 1$
$[i \geq n]$	c_3	m_3	1
$[A[i] \leq A[k]]$	c_4	m_4	$n - 1 - \ell$
$[A[i] > A[k]]$	c_5	m_5	ℓ
$k := i$	c_6	m_6	ℓ
$i := i + 1$	c_7	m_7	$n - 1$
$\mathit{result} := A[k]$	c_8	m_8	1

ℓ — the number of iterations of the cycle where $A[i] > A[k]$
(obviously $0 \leq \ell < n$)

Time Complexity of an Algorithm

By assigning values to

$$t(w) \;=\; c_0 m_0 + c_1 m_1 + \;\cdots\; + c_8 m_8,$$

we obtain

$$t(w) \;=\; d_1 \;+\; d_2 \cdot (n-1) \;+\; d_3 \cdot (n-1-\ell) \;+\; d_4 \cdot \ell,$$

where

$$\begin{array}{lll} d_1 = c_0 + c_1 + c_3 + c_8 & \qquad & d_3 = c_4 \\ d_2 = c_2 + c_7 & & d_4 = c_5 + c_6 \end{array}$$

After simplification we have

$$t(w) \;=\; (d_2 + d_3) \cdot n \;+\; (d_4 - d_3) \cdot \ell \;+\; (d_1 - d_2 - d_3)$$

Remark: $t(w)$ is not the time complexity but the running time for a particular input w

Time Complexity of an Algorithm

For example, if the execution times of instructions will be:

Instr.	symb.	time
$k := 0$	c_0	4
$i := 1$	c_1	4
$[i < n]$	c_2	10
$[i \geq n]$	c_3	12
$[A[i] \leq A[k]]$	c_4	14
$[A[i] > A[k]]$	c_5	12
$k := i$	c_6	5
$i := i + 1$	c_7	6
$\mathit{result} := A[k]$	c_8	5

then $d_1 = 25$, $d_2 = 16$, $d_3 = 14$, and $d_4 = 17$.

In this case is $t(w) = 30n + 3\ell - 5$.

For the input $w = ([3, 8, 4, 5, 2], 5)$ is $n = 5$ and $\ell = 1$, therefore $t(w) = 30 \cdot 5 + 3 \cdot 1 - 5 = 148$.

Time Complexity of an Algorithm

It can depend on details of implementation and on the precise values of constants, for which inputs of size n the compution takes the longest time (i.e., which are the worst cases):

The running time of algorithm $\textsc{Find-Max}$ for an input $w = (A, n)$ of size n:

$$t(w) \;=\; (d_2 + d_3) \cdot n \;+\; (d_4 - d_3) \cdot \ell \;+\; (d_1 - d_2 - d_3)$$

- If $d_3 \geq d_4$ — the worst cases are those where ℓ has the smallest value $\ell = 0$ — for example inputs of the form $[0, 0, \ldots, 0]$ or of the form $[n,\ n-1,\ n-2,\ \ldots,\ 2,\ 1]$
- If $d_3 \leq d_4$ — the worst are those cases where ℓ has the greatest value $\ell = n - 1$ — for example inputs of the form $[0, 1, \ldots, n-1]$

Time Complexity of an Algorithm

The time complexity $T(n)$ of algorithm FIND-MAX in the worst case is given as follows:

- If $d_3 \geq d_4$:

$$T(n) \ = \ (d_2 + d_3) \cdot n + (d_1 - d_2 - d_3)$$

- If $d_3 \leq d_4$:

$$\begin{aligned} T(n) &= (d_2 + d_3) \cdot n \ + \ (d_4 - d_3) \cdot (n-1) \ + \ (d_1 - d_2 - d_3) \\ &= (d_2 + d_4) \cdot n \ + \ (d_1 - d_2 - d_4) \end{aligned}$$

Example: For $d_1 = 25$, $d_2 = 16$, $d_3 = 14$, $d_4 = 17$ is

$$\begin{aligned} T(n) &= (16 + 17) \cdot n \ + \ (25 - 16 - 17) \\ &= 33n - 8 \end{aligned}$$

Time Complexity of an Algorithm

In both cases (when $d_3 \geq d_4$ or when $d_3 \leq d_4$), the time complexity of the algorithm Find-Max is a function

$$T(n) = an + b$$

where a and b are some constants whose precise values depend on the execution time of individual instructions.

Remark: These constants could be expressed as

$$a = d_2 + \max\{d_3, d_4\} \qquad\qquad b = d_1 - d_2 - \max\{d_3, d_4\}$$

For example

$$T(n) = 33n - 8$$

Time Complexity of an Algorithm

If it would be sufficient to find out that the time complexity of the algorithm Find-Max is some function of the form

$$T(n) = an + b,$$

where the precise values of constants a and b would not be important for us, the whole analysis could be considerably simpler.

- In fact, we usually do not want to know precisely how function $T(n)$ look (in general, it can be a very complicated function), and it would be sufficient to know that values of the function $T(n)$ "approximately" correspond to values of a function $S(n) = an + b$, where a and b are some constants.

Time Complexity of an Algorithm

For a given function $T(n)$ expressing the time or space complexity, it is usually sufficient to express it approximately — to have an **estimation** where

- we ignore the less important parts
  (e.g., in function $T(n) = 15n^2 + 40n - 5$ we can ignore $40n$ and -5, and to consider function $T(n) = 15n^2$ instead of the original function),
- we ignore multiplication constants
  (e.g., instead of function $T(n) = 15n^2$ we will consider function $T(n) = n^2$)
- we won't ignore constants in exponents — for example there is a big difference between functions $T_1(n) = n^2$ and $T_2(n) = n^3$.
- we will be interested how function $T(n)$ behaves for "big" values of n, we can ignore its behaviour on small values

Growth of Functions

A program works on an input of size n.
Let us assume that for an input of size n, the program performs $T(n)$ operations and that an execution of one operation takes $1\,\mu\mathrm{s}$ ($10^{-6}\,\mathrm{s}$).

	n							
$T(n)$	20	40	60	80	100	200	500	1000
n	20 µs	40 µs	60 µs	80 µs	0.1 ms	0.2 ms	0.5 ms	1 ms
$n \log n$	86 µs	0.213 ms	0.354 ms	0.506 ms	0.664 ms	1.528 ms	4.48 ms	9.96 ms
n^2	0.4 ms	1.6 ms	3.6 ms	6.4 ms	10 ms	40 ms	0.25 s	1 s
n^3	8 ms	64 ms	0.216 s	0.512 s	1 s	8 s	125 s	16.7 min.
n^4	0.16 s	2.56 s	12.96 s	42 s	100 s	26.6 min.	17.36 hours	11.57 days
2^n	1.05 s	12.75 days	36560 years	$38.3\cdot10^{9}$ years	$40.1\cdot10^{15}$ years	$50\cdot10^{45}$ years	$10.4\cdot10^{136}$ years	–
$n!$	77147 years	$2.59\cdot10^{34}$ years	$2.64\cdot10^{68}$ years	$2.27\cdot10^{105}$ years	$2.96\cdot10^{144}$ years	–	–	–

Growth of Functions

Let us consider 3 algorithms with complexities $T_1(n) = n$, $T_2(n) = n^3$, $T_3(n) = 2^n$. Our computer can do in a reasonable time (the time we are willing to wait) 10^{12} steps.

Complexity	Input size
$T_1(n) = n$	10^{12}
$T_2(n) = n^3$	10^4
$T_3(n) = 2^n$	40

Growth of Functions

Let us consider 3 algorithms with complexities $T_1(n) = n$, $T_2(n) = n^3$, $T_3(n) = 2^n$. Our computer can do in a reasonable time (the time we are willing to wait) 10^{12} steps.

Complexity	Input size
$T_1(n) = n$	10^{12}
$T_2(n) = n^3$	10^4
$T_3(n) = 2^n$	40

Now we speed up our computer 1000 times, meaning it can do 10^{15} steps.

Complexity	Input size	Growth
$T_1(n) = n$	10^{15}	$1000\times$
$T_2(n) = n^3$	10^5	$10\times$
$T_3(n) = 2^n$	50	$+10$

Asymptotic Notation

In the following, we will consider functions of the form $f : \mathbb{N} \to \mathbb{R}$, where:

- The values of $f(n)$ need not to be defined for all values of $n \in \mathbb{N}$ but there must exist some constant n_0 such that the value of $f(n)$ is defined for all $n \in \mathbb{N}$ such that $n \geq n_0$.

  Example: Function $f(n) = \log_2(n)$ is not defined for $n = 0$ but it is defined for all $n \geq 1$.

- There must exist a constant n_0 such that for all $n \in \mathbb{N}$, where $n \geq n_0$, is $f(n) \geq 0$.

  Example: It holds for function $f(n) = n^2 - 25$ that $f(n) \geq 0$ for all $n \geq 5$.

Asymptotic Notation

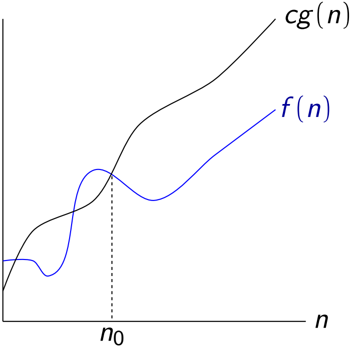
Let us take an arbitrary function $f : \mathbb{N} \rightarrow \mathbb{R}$. Expressions $O(f)$, $\Omega(f)$, and $\Theta(f)$ denote **sets of functions** of the type $\mathbb{N} \rightarrow \mathbb{R}$, where:

- $O(f)$ – the set of all functions that grow at most as fast as f
- $\Omega(f)$ – the set of all functions that grow at least as fast as f
- $\Theta(f)$ – the set of all functions that grow as fast as f

Remark: These are not definitions! The definitions will follow on the next slides.

- O – big "O"
- Ω – uppercase Greek letter "omega"
- Θ – uppercase Greek letter "theta"

Asymptotic Notation – Symbol O



Definition

Let us consider an arbitrary function $g : \mathbb{N} \rightarrow \mathbb{R}$. For a function $f : \mathbb{N} \rightarrow \mathbb{R}$ we have $f \in O(g)$ iff

$$(\exists c > 0)(\exists n_0 \geq 0)(\forall n \geq n_0)\big(f(n) \leq c\,g(n)\big).$$

Asymptotic Notation – Symbol O

Remarks:

- c is a posive real number (i.e., $c \in \mathbb{R}$ and $c > 0$)
- n_0 and n are natural numbers (i.e., $n_0 \in \mathbb{N}$ and $n \in \mathbb{N}$)

Asymptotic Notation – Symbol O

Example: Let us consider functions $f(n) = 2n^2 + 3n + 7$ and $g(n) = n^2$.

We want to show that $f \in O(g)$, i.e., $f \in O(n^2)$:

- Approach 1:

  Let us take for example $c = 3$.

  $$cg(n) = 3n^2 = 2n^2 + \frac{1}{2}n^2 + \frac{1}{2}n^2$$

  We need to find some n_0 such that for all $n \geq n_0$ it holds that

  $$2n^2 \geq 2n^2 \qquad \frac{1}{2}n^2 \geq 3n \qquad \frac{1}{2}n^2 \geq 7$$

  We can easily check that for example $n_0 = 6$ satisfies this.

  For each $n \geq 6$ we have $cg(n) \geq f(n)$:

  $$cg(n) = 3n^2 = 2n^2 + \frac{1}{2}n^2 + \frac{1}{2}n^2 \geq 2n^2 + 3n + 7 = f(n)$$

Asymptotic Notation – Symbol O

The example where $f(n) = 2n^2 + 3n + 7$ and $g(n) = n^2$:

- Approach 2:

  Let us take $c = 12$.

  $$cg(n) = 12n^2 = 2n^2 + 3n^2 + 7n^2$$

  We need to find some n_0 such that for all $n \geq n_0$ we have

  $$2n^2 \geq 2n^2 \qquad 3n^2 \geq 3n \qquad 7n^2 \geq 7$$

  These inequalities obviously hold for $n_0 = 1$, and so for each $n \geq 1$ we have $cg(n) \geq f(n)$:

  $$cg(n) = 12n^2 = 2n^2 + 3n^2 + 7n^2 \geq 2n^2 + 3n + 7 = f(n)$$

Asymptotic Notation – Symbol O

Proposition

Let us assume that a and b are constants such that $a > 0$ and $b > 0$, and k and ℓ are some arbitrary constants where $k \geq 0$, $\ell \geq 0$ and $k < \ell$.

Let us consider functions

$$f(n) = a \cdot n^k \qquad\qquad g(n) = b \cdot n^\ell$$

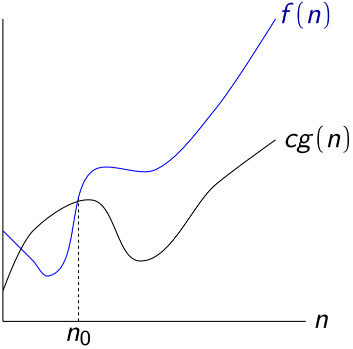
For each such functions f and g it holds that $f \in O(g)$:

Proof: Let us take $c = \frac{a}{b}$.

Because for $n \geq 1$ we obviously have $n^k \leq n^\ell$ (since $k \leq l$), for $n \geq 1$ we have

$$c \cdot g(n) = \frac{a}{b} \cdot g(n) = \frac{a}{b} \cdot b \cdot n^\ell = a \cdot n^\ell \geq a \cdot n^k = f(n)$$

Asymptotic Notation – Symbol Ω



Definition

Let us consider an arbitrary function $g : \mathbb{N} \rightarrow \mathbb{R}$. For a function $f : \mathbb{N} \rightarrow \mathbb{R}$ we have $f \in \Omega(g)$ iff

$$(\exists c > 0)(\exists n_0 \geq 0)(\forall n \geq n_0)\big(c\,g(n) \leq f(n)\big).$$

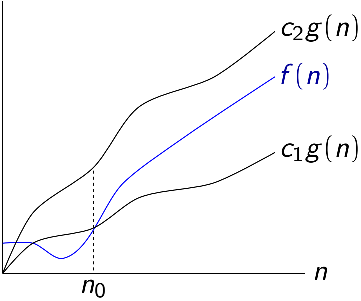
Asymptotic Notation – Symbol Ω

It is not difficult to prove the following proposition:

For arbitrary functions f and g we have:

$$f \in O(g) \qquad \text{iff} \qquad g \in \Omega(f)$$

Asymptotic Notation – Symbol Θ



Definition

Let us consider an arbitrary function $g : \mathbb{N} \to \mathbb{R}$. For a function $f : \mathbb{N} \to \mathbb{R}$ we have $f \in \Theta(g)$ iff

$$(\exists c_1 > 0)(\exists c_2 > 0)(\exists n_0 \geq 0)(\forall n \geq n_0)\big(c_1\, g(n) \leq f(n) \leq c_2\, g(n)\big).$$

Asymptotic Notation – Symbol Θ

For arbitrary functions f and g we have:

$$f \in \Theta(g) \qquad \text{iff} \qquad f \in O(g) \text{ a } f \in \Omega(g)$$

$$f \in \Theta(g) \qquad \text{iff} \qquad f \in O(g) \text{ a } g \in O(f)$$

$$f \in \Theta(g) \qquad \text{iff} \qquad g \in \Theta(f)$$

Asymptotic Notation

For arbitrary functions f, g, and h we have:

- if $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$
- if $f \in \Omega(g)$ and $g \in \Omega(h)$ then $f \in \Omega(h)$
- if $f \in \Theta(g)$ and $g \in \Theta(h)$ then $f \in \Theta(h)$

Asymptotic Notation

Examples:

$n \in O(n^2)$

$1000n \in O(n)$

$2^{\log_2 n} \in \Theta(n)$

$n^3 \notin O(n^2)$

$n^2 \notin O(n)$

$n^3 + 2^n \notin O(n^2)$

$n^3 \in O(n^4)$

$0.00001n^2 - 10^{10}n \in \Theta(10^{10}n^2)$

$n^3 - n^2 \log_2^3 n + 1000n - 10^{100} \in \Theta(n^3)$

$n^3 + 1000n - 10^{100} \in O(n^3)$

$n^3 + n^2 \notin \Theta(n^2)$

$n! \notin O(2^n)$

Asymptotic Notation

- There are pairs of functions f and g such that

$$f \notin O(g) \qquad \text{and} \qquad g \notin O(f),$$

for example

$$f(n) = n \qquad\qquad g(n) = n^{1+\sin(n)}.$$

- $O(1)$ denotes the set of all **bounded** functions, i.e., functions whose function values can be bounded from above by a constant.

Asymptotic Notation

- For any pair of functions f, g we have:
  - $\max(f, g) \in \Theta(f + g)$
  - if $f \in O(g)$ then $f + g \in \Theta(g)$

- For any functions f_1, f_2, g_1, g_2 we have:
  - if $f_1 \in O(f_2)$ and $g_1 \in O(g_2)$ then $f_1 + g_1 \in O(f_2 + g_2)$ and $f_1 \cdot g_1 \in O(f_2 \cdot g_2)$
  - if $f_1 \in \Theta(f_2)$ and $g_1 \in \Theta(g_2)$ then $f_1 + g_1 \in \Theta(f_2 + g_2)$ and $f_1 \cdot g_1 \in \Theta(f_2 \cdot g_2)$

Asymptotic Notation

- A function f is called:

  logarithmic, if $f(n) \in \Theta(\log n)$
  linear, if $f(n) \in \Theta(n)$
  quadratic, if $f(n) \in \Theta(n^2)$
  cubic, if $f(n) \in \Theta(n^3)$
  polynomial, if $f(n) \in O(n^k)$ for some $k > 0$
  exponential, if $f(n) \in O(c^{n^k})$ for some $c > 1$ and $k > 0$

- Exponential functions are often written in the form $2^{O(n^k)}$ when the asymptotic notation is used, since then we do not need to consider different bases.

Asymptotic Notation

As mentioned before, expressions $O(g)$, $\Omega(g)$, and $\Theta(g)$ denote certain sets of functions.

In some texts, these expressions are sometimes used with a slightly different meaning:

- an expression $O(g)$, $\Omega(g)$ or $\Theta(g)$ does not represent the corresponding set of functions but **some** function from this set.

This convention is often used in equations and inequations.

Example: $3n^3 + 5n^2 - 11n + 2 = 3n^3 + O(n^2)$

When using this convention, we can for example write $f = O(g)$ instead of $f \in O(g)$.

Complexity of Algorithms

Let us say we would like to analyze the time complexity $T(n)$ of some algorithm consisting of instructions $I_1, I_2, \ldots, I_k$:

- If $m_1, m_2, \ldots, m_k$ are the numbers of executions of individual instructions for some input w (i.e., the instruction I_i is performed m_i times for the input w), then the total number of executed instructions for input w is

$$T(n) = c_1 m_1 + c_2 m_2 + \cdots + c_k m_k .$$

- Let us consider functions $f_1, f_2, \ldots, f_k$, where $f_i : \mathbb{N} \to \mathbb{R}$, and where $f_i(n)$ is the maximum of numbers of executions of instruction I_i for all inputs of size n.
- Obviously, $T \in \Omega(f_i)$ for any function f_i.
- It is also obvious that $T \in O(f_1 + f_2 + \cdots + f_k)$.

Complexity of Algorithms

- Let us recall that if $f \in O(g)$ then $f + g \in O(g)$.
- If there is a function f_i such that for all f_j, where $j \neq i$, we have $f_j \in O(f_i)$, then
$$T \in O(f_i).$$
- This means that in an analysis of the time complexity $T(n)$, we can restrict our attention to the number of executions of the instruction that is performed most frequently (and which is performed at most $f_i(n)$ times for an input of size n), since we have
$$T \in \Theta(f_i).$$

Complexity of Algorithms

Example: In the analysis of the complexity of the searching of a number in a sequence we obtained

$$f(n) = an + b\,.$$

If we would not like to do such a detailed analysis, we could deduce that the time complexity of the algorithm is $\Theta(n)$, because:

- The algorithm contains only one cycle, which is performed $(n-1)$ times for an input of size n, the number of iterations of the cycle is in $\Theta(n)$.
- Several instructions are performed in one iteration of the cycle. The number of these instructions is bounded from both above and below by some constant independent on the size of the input.
- Other instructions are performed at most once, and so they contribute to the total running time by adding a constant.

Complexity of Algorithms

Let us try to analyze the time complexity of the following algorithm:

Algorithm 2: Insertion sort

```
INSERTION-SORT ($A$, $n$):
begin
    for $j := 1$ to $n - 1$ do
        $x := A[j]$
        $i := j - 1$
        while $i \geq 0$ and $A[i] > x$ do
            $A[i + 1] := A[i]$
            $i := i - 1$
        end
        $A[i + 1] := x$
    end
end
```

I.e., we want to find a function $T(n)$ such that the time complexity of the algorithm INSERTION-SORT in the worst case is in $\Theta(T(n))$.

Complexity of Algorithms

Let us consider inputs of size n:

- The outer cycle **for** is performed at most $n - 1$ times.
- The inner cycle **while** is performed at most $(j - 1)$ times for a given value j.
- There are inputs such that the cycle **while** is performed exactly $(j - 1)$ times for each value j from 2 to n.
- So in the worst case, the cycle **while** is performed exactly m times, where

$$m = 1 + 2 + \cdots + (n - 1) = (1 + (n - 1)) \cdot \tfrac{n-1}{2} = \tfrac{1}{2}n^2 - \tfrac{1}{2}n$$

- This means that the total running time of the algorithm Insertion-Sort in the worst case is $\Theta(n^2)$.

Complexity of Algorithms

In the previous case, we have computed the total number of executions of the cycle **while** accurately.

This is not always possible in general, or it can be quite complicated. It is also not necessary, if we only want an asymptotic estimation.

Complexity of Algorithms

For example, if we were not able to compute the sum of the arithmetic progression, we could proceed as follows:

- The outer cycle **for** is not performed more than n times and the inner cycle **while** is performed at most n times in each iteration of the outer cycle.

  So we have $T \in O(n^2)$.

- For some inputs, the cycle **while** is performed at least $\lceil n/2 \rceil$ times in the last $\lfloor n/2 \rfloor$ iterations of the cycle **for**.

  So the cycle **while** is performed at least $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ times for some inputs.

  $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil \geq (n/2 - 1) \cdot (n/2) = \frac{1}{4}n^2 - \frac{1}{2}n$

  This implies $T \in \Omega(n^2)$.

Space Complexity of Algorithms

- So far we have considered only the time necessary for a computation
- Sometimes the size of the memory necessary for the computation is more critical.

The **amount of memory** used by machine $\mathcal{M}$ in a computation on input w can be for example:

- the maximal number of bits necessary for storing all data for each configuration
- the maximal number of memory cells used during the computation

Definition

A **space complexity** of algorithm Alg running on machine $\mathcal{M}$ is the function $S : \mathbb{N} \to \mathbb{N}$, where $S(n)$ is the maximal amount of memory used by $\mathcal{M}$ for inputs of size n.

Space Complexity of Algorithms

- There can be two algorithms for a particular problem such that one of them has a smaller time complexity and the other a smaller space complexity.

- If the time-complexity of an algorithm is in $O(f(n))$ then also the space complexity is in $O(f(n))$ (note that the number of memory cells used in one instruction is bounded by some constant that does not depend on the size of an input).

- The space complexity can be much smaller than the time complexity — the space complexity of Insertion-Sort is $\Theta(n)$, while its time complexity is $\Theta(n^2)$.

Complexity of Algorithms

Some typical values of the size of an input n, for which an algorithm with the given time complexity usually computes the output on a "common PC" within a fraction of a second or at most in seconds.

(Of course, this depends on particular details. Moreover, it is assumed here that no big constants are hidden in the asymptotic notation)

$O(n)$	$O(n \log n)$	$O(n^2)$	$O(n^3)$
$1\,000\,000 - 100\,000\,000$	$100\,000 - 1\,000\,000$	$1000 - 10\,000$	$100 - 1000$

$2^{O(n)}$	$O(n!)$
$20 - 30$	$10 - 15$

Complexity of Algorithms

When we use asymptotic estimations of the complexity of algorithms, we should be aware of some issues:

- Asymptotic estimations describe only how the running time grows with the growing size of input instance.
- They do not say anything about exact running time. Some big constants can be hidden in the asymptotic notation.
- An algorithm with better asymptotic complexity than some other algorithm can be in reality faster only for very big inputs.
- We usually analyze the time complexity in the worst case. For some algorithms, the running time in the worst case can be much higher than the running time on “typical” instances.

Complexity of Algorithms

- This can be illustrated on algorithms for sorting.

Algorithm	Worst-case	Average-case
Bubblesort	$\Theta(n^2)$	$\Theta(n^2)$
Heapsort	$\Theta(n \log n)$	$\Theta(n \log n)$
Quicksort	$\Theta(n^2)$	$\Theta(n \log n)$

- Quicksort has a worse asymptotic complexity in the worst case than Heapsort and the same asymptotic complexity in an average case but it is usually faster in practice.

Complexity of Algorithms

Polynomial — an expression of the form

$$a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0$$

where $a_0, a_1, \ldots, a_k$ are constants.

Examples of polynomials:

$$4n^3 - 2n^2 + 8n + 13 \qquad 2n + 1 \qquad n^{100}$$

Function f is called **polynomial** if it is bounded from above by some polynomial, i.e., if there exists a constant k such that $f \in O(n^k)$.

For example, the functions belonging to the following classes are polynomial:

$$O(n) \qquad O(n \log n) \qquad O(n^2) \qquad O(n^5) \qquad O(\sqrt{n}) \qquad O(n^{100})$$

Complexity of Algorithms

Function such as 2^n or $n!$ are not polynomial — for arbitrarily big constant k we have

$$2^n \in \Omega(n^k) \qquad\qquad n! \in \Omega(n^k)$$

Polynomial algorithm — an algorithm whose time complexity is polynomial (i.e., bounded from above by some polynomial)

Roughly we can say that:

- polynomial algorithms are effiecient algorithms that can be used in practice for inputs of considerable size
- algorithms, which are not polynomial, can be used in practice only for rather small inputs

Complexity of Algorithms

The division of algorithms on polynomial and non-polynomial is very rough — we cannot claim that polynomial algorithms are always efficient and non-polynomial algorithms are not:

- an algorithm with the time complexity $\Theta(n^{100})$ is probably not very useful in practice,
- some algorithms, which are non-polynomial, can still work very efficiently for majority of inputs, and can have a time complexity bigger than polynomial only due to some problematic inputs, on which the computation takes long time.

Remark: Polynomial algorithms where the constant in the exponent is some big number (e.g., algorithms with complexity $\Theta(n^{100})$) almost never occur in practice as solutions of usual algorithmic problems.

Complexity of Algorithms

For most of common algorithmic problems, one of the following three possibilities happens:

- A polynomial algorithm with time complexity $O(n^k)$ is known, where k is some very small number (e.g., 5 or more often 3 or less).
- No polynomial algorithm is known and the best known algorithms have complexities such as $2^{\Theta(n)}$, $\Theta(n!)$, or some even bigger.

  In some cases, a proof is known that there does not exist a polynomial algorithm for the given problem (it cannot be constructed).
- No algorithm solving the given problem is known (and it is possibly proved that there does not exist such algorithm)

Complexity of Algorithms

A typical example of polynomial algorithm — matrix multiplication with time complexity $\Theta(n^3)$ and space complexity $\Theta(n^2)$:

Algorithm 3: Matrix multiplication

```
Matrix-Mult $(A, B, C, n)$:
begin
    for $i := 1$ to $n$ do
        for $j := 1$ to $n$ do
            $x := 0$
            for $k := 1$ to $n$ do
                $x := x + A[i][k] * B[k][j]$
            end
            $C[i][j] := x$
        end
    end
end
```

Complexity of Algorithms

- For a rough estimation of complexity, it is often sufficient to count the number of nested loops — this number then gives the degree of the polynomial

  Example: Three nested loops in the matrix multiplication — the time complexity of the algorithm is $O(n^3)$.

- If it is not the case that all the loops go from 0 to n but the number of iterations of inner loops are different for different iterations of an outer loops, a more precise analysis can be more complicated.

  It is often the case, that the sum of some sequence (e.g., the sum of arithmetic or geometric progression) is then computed in the analysis.

  The results of such more detailed analysis often does not differ from the results of a rough analysis but in many cases the time complexity resulting from a more detailed analysis can be considerably smaller than the time complexity following from the rough analysis.

Arithmetic progression — a sequence of numbers $a_0, a_1, \ldots, a_{n-1}$, where

$$a_i = a_0 + i \cdot d,$$

where d is some constant independent on i.

Remark: So in an arithmetic progression, we have $a_{i+1} = a_i + d$ for each i.

The sum of an arithmetic progression:

$$\sum_{i=0}^{n-1} a_i \;=\; a_0 + a_1 + \cdots + a_{n-1} \;=\; \frac{1}{2} n \left(a_{n-1} + a_0 \right)$$

Complexity of Algorithms

Example:

$$1 + 2 + \cdots + n \;=\; \frac{1}{2}n(n+1) \;=\; \frac{1}{2}n^2 + \frac{1}{2}n \;=\; \Theta(n^2)$$

For example, for $n = 100$ we have

$$1 + 2 + \cdots + 100 \;=\; 50 \cdot 101 \;=\; 5050.$$

Remark: To see this, we can note that

$$1 + 2 + \cdots + 100 \;=\; (1 + 100) + (2 + 99) + \cdots + (50 + 51),$$

where we compute the sum of 50 pairs of number, where the sum of each pair is 101.

Geometric progression — a sequence of numbers $a_0, a_1, \ldots, a_n$, where

$$a_i = a_0 \cdot q^i,$$

where q is some constant independent on i.

Remark: So in a geometric progression we have $a_{i+1} = a_i \cdot q$.

The sum of a geometic progression (where $q \neq 1$):

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + \cdots + a_n \; = \; a_0 \frac{q^{n+1} - 1}{q - 1}$$

Complexity of Algorithms

Example:

$$1 + q + q^2 + \cdots + q^n \;=\; \frac{q^{n+1} - 1}{q - 1}$$

In particular, for $q = 2$:

$$1 + 2^1 + 2^2 + 2^3 + \cdots + 2^n \;=\; \frac{2^{n+1} - 1}{2 - 1} \;=\; 2 \cdot 2^n - 1 \;=\; \Theta(2^n)$$

Complexity of Algorithms

An **exponential** function: a function of the form c^n, where c is a constant — e.g., function 2^n

Logarithm — the inverse function to an exponential function: for a given n,

$$\log_c n$$

is the value x such that $c^x = n$.

Complexity of Algorithms

n	2^n
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048
12	4096
13	8192
14	16384
15	32768
16	65536
17	131072
18	262144
19	524288
20	1048576

n	$\lceil \log_2 n \rceil$
0	—
1	0
2	1
3	2
4	2
5	3
6	3
7	3
8	3
9	4
10	4
11	4
12	4
13	4
14	4
15	4
16	4
17	5
18	5
19	5
20	5

n	$\log_2 n$
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
2048	11
4096	12
8192	13
16384	14
32768	15
65536	16
131072	17
262144	18
524288	19
1048576	20

Complexity of Algorithms

Proposition

For any $a, b > 1$ and any $n > 0$ we have

$$\log_a n = \frac{\log_b n}{\log_b a}$$

Proof: From $n = a^{\log_a n}$ it follows that $\log_b n = \log_b(a^{\log_a n})$.
Since $\log_b(a^{\log_a n}) = \log_a n \cdot \log_b a$, we obtain $\log_b n = \log_a n \cdot \log_b a$, from which the above mentioned conclusion follows directly. □

Due to this observation, the base of a logarithm is often omited in the asymptotic notation: for example, instead of $\Theta(n \log_2 n)$ we can write $\Theta(n \log n)$.

Complexity of Algorithms

Examples where exponential functions and logarithms can appear in an analysis of algorithms:

- Some value is repeatedly decreased to one half or is repeatedly doubled.

  For example, in the **binary search**, the size of an interval halves in every iteration of the loop.

  Let us assume that an array has size n.

  What is the minimal size of an array n, for which the algorithm performs at least k iterations?

  The answer: 2^k

  So we have $k = \log_2(n)$. The time complexity of the algorithm is then $\Theta(\log n)$.

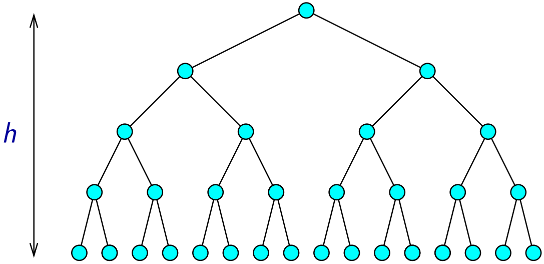
Complexity of Algorithms

- Using n bits we can represent numbers from 0 to $2^n - 1$.
- The minimal numbers of bits, which are sufficient for representing a natural number x in binary is
$$\lceil \log_2(x+1) \rceil .$$
- A perfectly balanced tree of height h has $2^{h+1} - 1$ nodes, and 2^h of these nodes are leaves.
- The height of a perfectly balanced binary tree with n nodes is $\log_2 n$.

  An illustrating example: If we would draw a balanced tree with $n = 1\,000\,000$ nodes in such a way that the distance between neighbouring nodes would be 1 cm and the height of each layer of nodes would be also 1 cm, the width of the tree would be 10 km and its height would be approximately 20 cm.

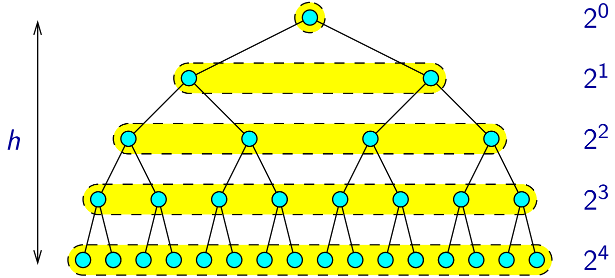
Complexity of Algorithms

A perfectly balanced binary tree of height h:



Complexity of Algorithms

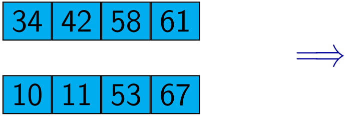
A perfectly balanced binary tree of height h:



Complexity of Algorithms

Example: Algorithm MERGE-SORT.

The main idea of the algorithm: Two sorted sequences can be easily merged into one sorted sequence.
If both sequences have together n elements then this operation can be done in n steps.



34	42	58	61

$\Longrightarrow$

10	11	53	67

Complexity of Algorithms

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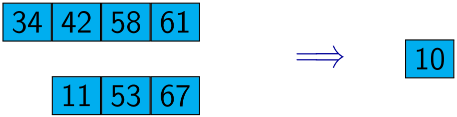

34 42 58 61

11 53 67

⟹

10

Complexity of Algorithms

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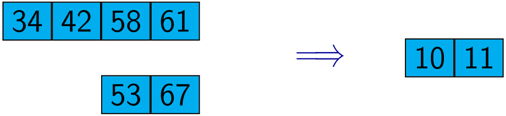

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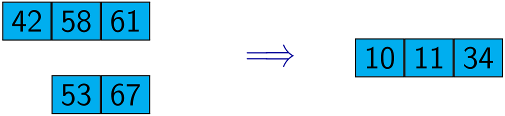

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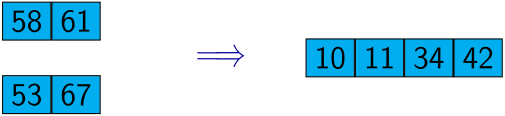

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Complexity of Algorithms

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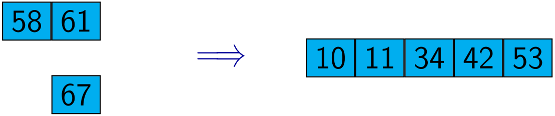

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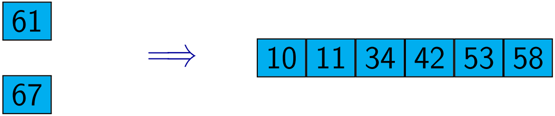

61

67

$\Longrightarrow$

10 | 11 | 34 | 42 | 53 | 58

Complexity of Algorithms

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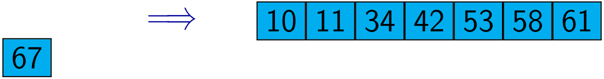

Complexity of Algorithms

Example: Algorithm MERGE-SORT.

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If both sequences have together n elements then this operation can be done in n steps.

$\Longrightarrow$

10	11	34	42	53	58	61	67

Complexity of Algorithms

Algorithm 4: Merge sort

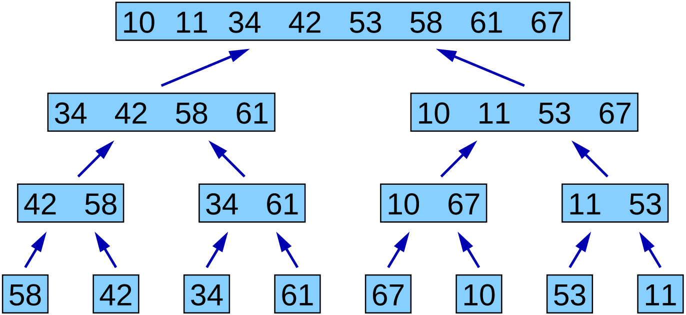
```
Merge-Sort ($A, p, r$):
begin
    if $r - p > 1$ then
        $q := \lfloor (p + r) / 2 \rfloor$
        Merge-Sort($A, p, q$)
        Merge-Sort($A, q, r$)
        Merge($A, p, q, r$)
    end
end
```

To sort an array A containing elements $A[0], A[1], \cdots, A[n-1]$ we call $\textsc{Merge-Sort}(A, 0, n)$.

Remark: Procedure $\textsc{Merge}(A, p, q, r)$ merges sorted sequences stored in $A[p \mathinner{..} q-1]$ and $A[q \mathinner{..} r-1]$ into one sequence stored in $A[p \mathinner{..} r-1]$.

Complexity of Algorithms

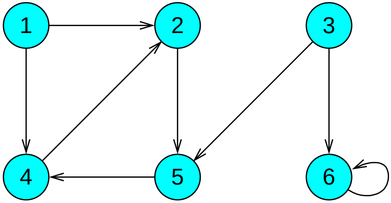
Input: 58, 42, 34, 61, 67, 10, 53, 11

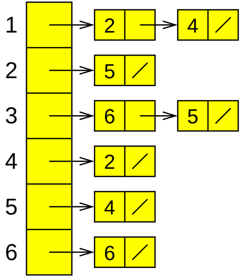


The tree of recursive calls has $\Theta(\log n)$ layers. On each layer, $\Theta(n)$ operations are performed. The time complexity of MERGE-SORT is $\Theta(n \log n)$.

Complexity of Algorithms

Representations of a graph:

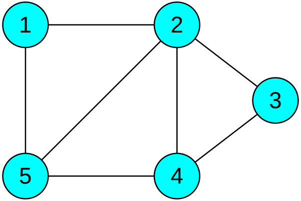


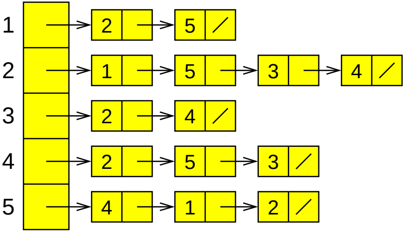


$$\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

Complexity of Algorithms

Representations of a graph:





	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Complexity of Algorithms

Finding the shortest path in a graph where edges are not weighted:

- Breadth-first search
- The input is a graph G (with a set of nodes V) and an initial node s.
- The algorithm finds the shortest paths from node s for all nodes.
- For a graph with n nodes and m edges, the running time of the algorithm is $\Theta(n+m)$.

Complexity of Algorithms

Algorithm 5: Breadth-first search

```
BFS ($G, s$):
begin
    BFS-INIT($G, s$)
    ENQUEUE($\mathcal{Q}, s$)
    while $\mathcal{Q} \neq \emptyset$ do
        $u :=$ DEQUEUE($\mathcal{Q}$)
        for each $v \in edges[u]$ do
            if $color[v] = $ WHITE then
                $color[v] :=$ GRAY
                $d[v] := d[u] + 1$
                $pred[v] := u$
                ENQUEUE($\mathcal{Q}, v$)
            end
        end
        $color[u] :=$ BLACK
    end
end
```

Complexity of Algorithms

Algorithm 6: Breadth-first search — initialization

```
BFS-INIT ($G, s$):
begin
    for each $u \in V - \{s\}$ do
        $color[u] :=$ WHITE
        $d[u] := \infty$
        $pred[u] :=$ NIL
    end
    $color[s] :=$ GRAY
    $d[u] := 0$
    $pred[u] :=$ NIL
    $\mathcal{Q} := \emptyset$
end
```

Complexity of Algorithms

- A set containing n elements has 2^n subsets.

  Consider for example an algorithm solving a given problem by **brute force** where it tests some property for each subset of a given set.

- If it would be sufficient to consider only subsets of some particular size k, the total number of these subsets is

  $$\binom{n}{k}$$

  For some values of k, the total number of these subsets is not much smaller than 2^n:

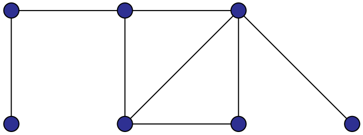
  It can be for example shown that

  $$\binom{n}{\lfloor n/2 \rfloor} \geq \frac{2^n}{n} .$$

Independent set (IS) problem

Input: An undirected graph G, a number k.

Question: Is there an independent set of size k in the graph G?



$k = 4$

Remark: An **independent set** in a graph is a subset of nodes of the graph such that no pair of nodes from this set is connected by an edge.

Complexity of Algorithms

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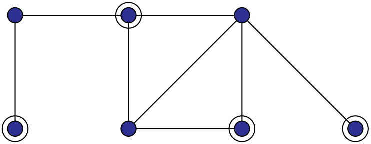
Input: An undirected graph G, a number k.

Question: Is there an independent set of size k in the graph G?

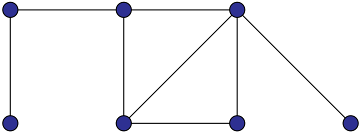


$k = 4$

Remark: An **independent set** in a graph is a subset of nodes of the graph such that no pair of nodes from this set is connected by an edge.

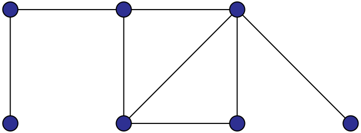
Complexity of Algorithms

An example of an instance where the answer is YES:



$k = 4$

An example of an instance where the answer is NO:



$k = 5$

Complexity of Algorithms

Let us have an algorithm solving the independent set problem by brute force in such a way that it tests for each subset with k elements of the set of nodes (with n nodes), if it forms an independent set. The time complexity of the algorithm is $2^{\Theta(n)}$.

Examples of Algorithmic Problems

The following slides contain examples of several algorithmic problems:

- these problems can be solved by algorithms — usually it is not difficult to find out an algorithm with an exponential time complexity solving the given problem
- no polynomial time algorithm is known for any of these problems
- on the other hand, for any of these problems it is not proved there cannot exist a polynomial time algorithm for the given problem
- all these problems are examples of so called **NP-complete** problems

Problem SAT

SAT (boolean satisfiability problem)

Input: A formula of propositional logic φ.

Question: Is φ satisfiable?

Example:

Formula $\varphi_1 = p \wedge (\neg q \vee r)$ is satisfiable: e.g., for valuation v where $v(p) = 1$, $v(q) = 0$, $v(r) = 1$, it holds that $v \models \varphi_1$.

Formula $\varphi_2 = (p \wedge \neg p) \vee (\neg q \wedge r \wedge q)$ is not satisfiable: for every valuation v it holds that $v \not\models \varphi_2$.

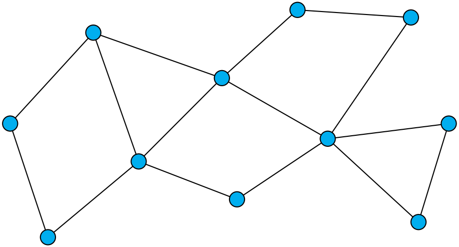
Graph Coloring

Graph coloring

Input: An undirected graph G, a natural number k.

Question: Is it possible to color nodes of the graph G using k colors in such a way that there is no pair of nodes where both nodes are colored with the same color and connected with an edge?

Example: $k = 3$



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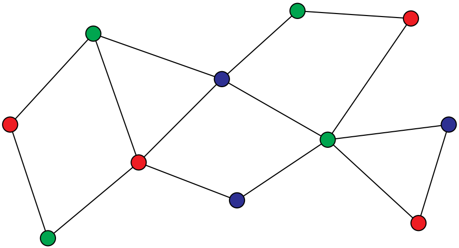

Answer: Yes

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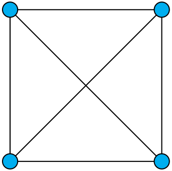

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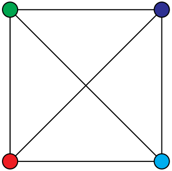


Answer: No

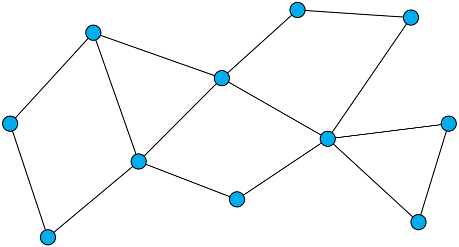
VC – Vertex Cover

VC – vertex cover

Input: An undirected graph G and a natural number k.

Question: Is there some subset of nodes of G of size k such that every edge has at least one of its nodes in this subset?

Example: $k = 6$



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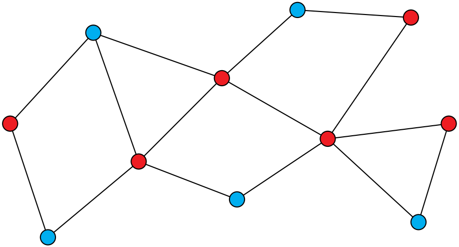


Answer: YES

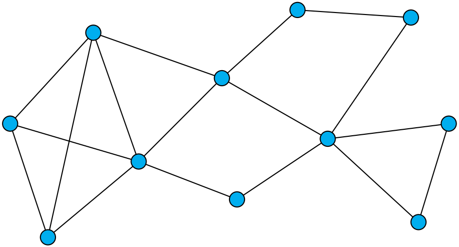
CLIQUE

CLIQUE

Input: An undirected graph G and a natural number k.

Question: Is there some subset of nodes of G of size k such that every two nodes from this subset are connected by an edge?

Example: $k = 4$



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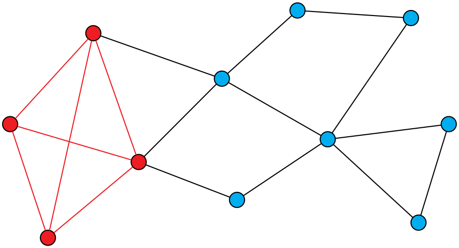


Answer: YES

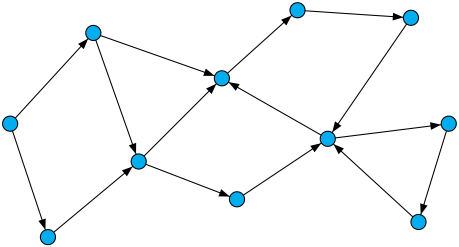
Hamiltonian Cycle

HC – Hamiltonian cycle

Input: A directed graph G.

Question: Is there a Hamiltonian cycle in G (i.e., a directed cycle going through each node exactly once)?

Example:



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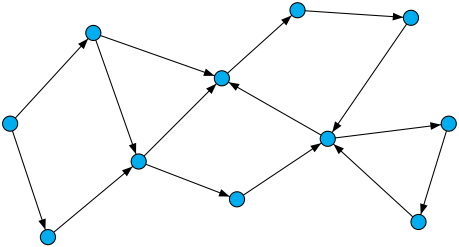

Answer: No

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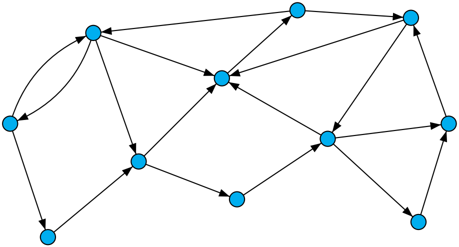

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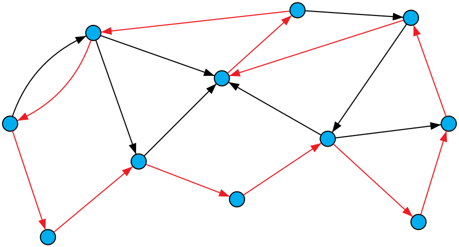


Answer: YES

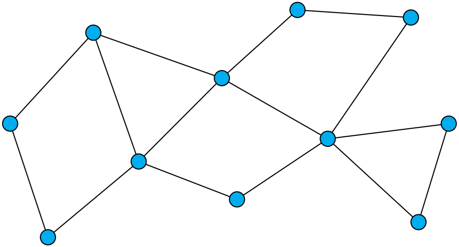
Hamiltonian Circuit

HK – Hamiltonian circuit

Input: An undirected graph G.

Question: Is there a Hamiltonian circuit in G (i.e., an undirected cycle going through each node exactly once)?

Example:



Answer: No

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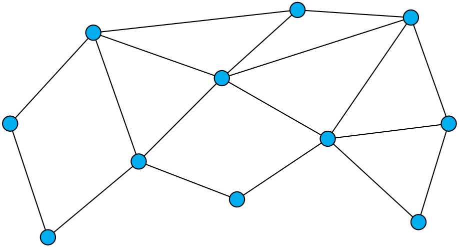

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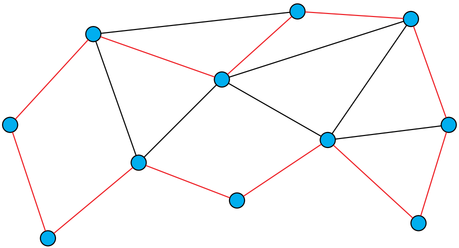


Answer: YES

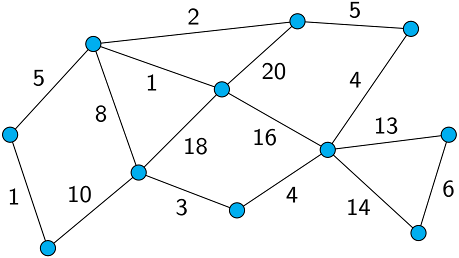
Traveling Salesman Problem

TSP - traveling salesman problem

Input: An undirected graph G with edges labelled with natural numbers and a number k.

Question: Is there a closed tour going through all nodes of the graph G such that the sum of labels of edges on this tour is at most k?

Example: $k = 70$



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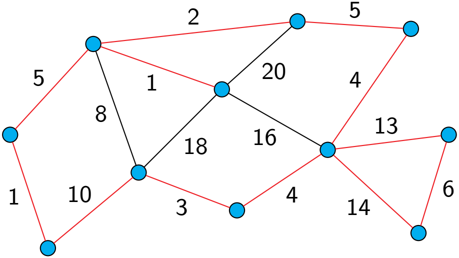


Answer: YES, since there is a tour with the sum 69.

SUBSET-SUM

Problem SUBSET-SUM

Input: A sequence $a_1, a_2, \ldots, a_n$ of natural numbers and a natural number s.

Question: Is there a set $I \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} a_i = s$?

In other words, the question is whether it is possible to select a subset with sum s of a given (multi)set of numbers.

Example: For the input consisting of numbers $3, 5, 2, 3, 7$ and number $s = 15$ the answer is $\textsc{Yes}$, since $3 + 5 + 7 = 15$.

For the input consisting of numbers $3, 5, 2, 3, 7$ and number $s = 16$ the answer is $\textsc{No}$, since no subset of these numbers has sum 16.

SUBSET-SUM

Remark:
The order of numbers $a_1, a_2, \ldots, a_n$ in an input is not important.

Note that this is not exactly the same as if we have formulated the problem so that the input is a set $\{a_1, a_2, \ldots, a_n\}$ and a number s — numbers cannot occur multiple times in a set but they can in a sequence.

SUBSET-SUM

Problem SUBSET-SUM is a special case of a **knapsack problem**:

Knapsack problem

Input: Sequence of pairs of natural numbers $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ and two natural numbers s and t.

Question: Is there a set $I \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} a_i \leq s$ and $\sum_{i \in I} b_i \geq t$?

SUBSET-SUM

Informally, the knapsack problem can be formulated as follows:

We have n objects, where the i-th object weights a_i grams and its price is b_i dollars.

The question is whether there is a subset of these objects with total weight at most s grams (s is the capacity of the knapsack) and with total price at least t dollars.

Remark:

Here we have formulated this problem as a decision problem.

This problem is usually formulated as an optimization problem where the aim is to find such a set $I \subseteq \{1, 2, \ldots, n\}$, where the value $\sum_{i \in I} b_i$ is maximal and where the condition $\sum_{i \in I} a_i \leq s$ is satisfied, i.e., where the capacity of the knapsack is not exceeded.

SUBSET-SUM

That SUBSET-SUM is a special case of the Knapsack problem can be seen from the following simple construction:

Let us say that $a_1, a_2, \ldots, a_n$, s_1 is an instance of SUBSET-SUM.
It is obvious that for the instance of the knapsack problem where we have the sequence (a_1, a_1), $(a_2, a_2), \ldots,$ (a_n, a_n), $s = s_1$ and $t = s_1$, the answer is the same as for the original instance of SUBSET-SUM.

SUBSET-SUM

If we want to study the complexity of problems such as SUBSET-SUM or the knapsack problem, we must clarify what we consider as the size of an instance.

Probably the most natural it is to define the size of an instance as the total number of bits needed for its representation.

We must specify how natural numbers in the input are represented – if in binary (resp. in some other numeral system with a base at least 2 (e.g., decimal or hexadecimal) or in unary.

- If we consider the total number of bits when numbers are written in **binary** as the size of an input, no polynomial time algorithm is known for SUBSET-SUM.
- If we consider the total number of bits when numbers are written in **unary** as the size of an input, SUBSET-SUM can be solved by an algorithm whose time complexity is polynomial.

ILP – Integer Linear Programming

Problem ILP (integer linear programming)

Input: An integer matrix A and an integer vector b.

Question: Is there an integer vector x such that $Ax \leq b$?

An example of an instance of the problem:

$$A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix}$$

So the question is if the following system of inequations has some integer solution:

$$\begin{array}{rcl} 3x_1 - 2x_2 + 5x_3 & \leq & 8 \\ x_1 + x_3 & \leq & -3 \\ 2x_1 + x_2 & \leq & 5 \end{array}$$

ILP – Integer Linear Programming

One of solutions of the system

$$\begin{array}{rcl} 3x_1 - 2x_2 + 5x_3 & \leq & 8 \\ x_1 + x_3 & \leq & -3 \\ 2x_1 + x_2 & \leq & 5 \end{array}$$

is for example $x_1 = -4$, $x_2 = 1$, $x_3 = 1$, i.e.,

$$x = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$$

because

$$\begin{array}{rcrcr} 3 \cdot (-4) - 2 \cdot 1 + 5 \cdot 1 & = & -9 & \leq & 8 \\ -4 + 1 & = & -3 & \leq & -3 \\ 2 \cdot (-4) + 1 & = & -7 & \leq & 5 \end{array}$$

So the answer for this instance is $\textsc{Yes}$.

ILP – Integer Linear Programming

Remark: A similar problem where the question for a given system of linear inequations is whether it has a solution in the set of **real** numbers, can be solved in a polynomial time.

Examples of Algorithmic Problems

The following slides constain several examples of typical algorithmic problems:

- **these problems cannot be solved algorithmically**
- for all of these problems, it is proved that there cannot exist an algorithm solving a given problem

Remark: Recall that an algorithm solves a given problem if for each input the algorithm halts after a finite number of steps and produces a correct output.

So it holds for the following problems that for each algorithm that would try to solve the given problem, we can find an example of an input such that either:

- the algorithm never halts, or
- it produces an incorrect output

Remark: In the case of decision problems, those problems, which are not algorithmically solvable, are called **undecidable**.

Příklady nerozhodnutelných problémů

We have already seen an example of an undecidable problem:

Problem

Input: Context-free grammars G_1 and G_2.

Question: Is $L(G_1) = L(G_2)$?

respectively

Problem

Input: A context-free grammar generating a language over an alphabet Σ.

Question: Is $L(G) = \Sigma^*$?

Other Undecidable Problems

Problem

Input: Context-free grammars G_1 and G_2.

Question: Is $L(G_1) \cap L(G_2) = \emptyset$?

Problem

Input: A context-free grammar G.

Question: Is G ambiguous?

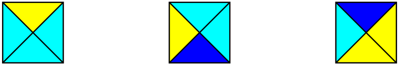
Other Undecidable Problems

Many problems dealing with a behaviour of programs are undecidable:

- Is for some input the output of a given program Yes?
- Does a given program halt for an arbitrary input?
- Do two given programs produce the same outputs for the same inputs?
- ...

Other Undecidable Problems

An input is a set of types of tiles, such as:

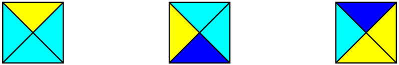


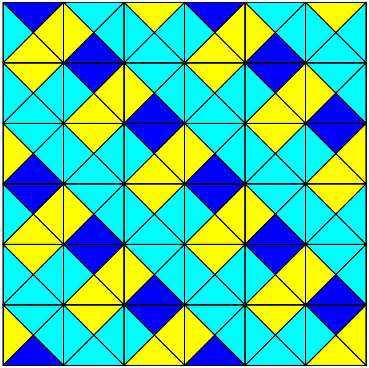
The question is whether it is possible to cover every finite area of an arbitrary size using the given types of tiles in such a way that the colors of neighboring tiles agree.

Remark: We can assume that we have an infinite number of tiles of all types.

The tiles cannot be rotated.

Other Undecidable Problems





Other Undecidable Problems

An input is a set of types of cards, such as:

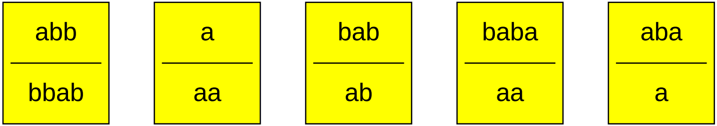
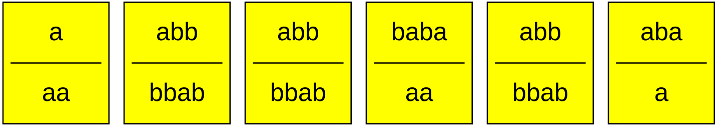


abb	a	bab	baba	aba
bbab	aa	ab	aa	a

The question is whether it is possible to construct from the given types of cards a non-empty finite sequence such that the concatenations of the words in the upper row and in the lower row are the same. Every type of a card can be used repeatedly.



a	abb	abb	baba	abb	aba
aa	bbab	bbab	aa	bbab	a

In the upper and in the lower row we have obtained the word `aabbabbbabaabbaba`.

Other Undecidable Problems

Problem

Input: A closed formula of predicate logic where $+$, $*$, and integer constants can be used as function symbols and $=$ and $<$ as predicate symbols.

Question: Is the given formula true in the domain of natural numbers (using the natural interpretation of all function and predicate symbols)?

An example of an input:

$$\forall x \exists y \forall z ((x * y = z) \wedge (y + 5 = x))$$

Remark: There is a close connection with Gödel's incompleteness theorem.

Other Undecidable Problems

It is interesting that an analogous problem, where **real** numbers are considered instead of natural numbers, is decidable (but the algorithm for it and the proof of its correctness are quite nontrivial).

Also when we consider natural numbers or integers and the same formulas as in the previous case but with the restriction that it is not allowed to use the multiplication function symbol $*$, the problem is algorithmically decidable.

Other Undecidable Problems

If the function symbol $*$ can be used then even the very restricted case is undecidable:

Hilbert's tenth problem

Input: A polynomial $f(x_1, x_2, \ldots, x_n)$ constructed from variables $x_1, x_2, \ldots, x_n$ and integer constants.

Question: Are there some natural numbers $x_1, x_2, \ldots, x_n$ such that $f(x_1, x_2, \ldots, x_n) = 0$?

An example of an input: $5x^2y - 8yz + 3z^2 - 15$

I.e., the question is whether

$$\exists x \exists y \exists z (5 * x * x * y + (-8) * y * z + 3 * z * z + (-15) = 0)$$

holds in the domain of natural numbers.

Other Undecidable Problems

Also the following problem is algorithmically undecidable:

Problem

Input: A closed formula φ of predicate logic.

Question: Is $\models \varphi$?

Remark: Notation $\models \varphi$ denotes that formula φ is logically valid, i.e., it is true in all interpretations.