Tautologies

Definition

Formula φ is a **tautology** if $v \models \varphi$ holds for every truth valuation v (i.e., if φ is true in every valuation).

Example: *"If it is raining, then it is raining."*

$$p \rightarrow p$$

Example: *"It is Friday today, or it is not Friday today."*

$$q \vee \neg q$$

Tautologies

An example of a more complicated tautology:

$$(p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$$

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$\neg p$	$(p \rightarrow \neg q) \rightarrow \neg p$	φ
0	0	1	1	1	1	1	1
0	1	1	0	1	1	1	1
1	0	0	1	1	0	0	1
1	1	1	0	0	0	1	1

Tautologies

Quite important are tautologies of the form $\varphi \rightarrow \psi$ or $\varphi \leftrightarrow \psi$ — they can be used for logical inference:

- If $\varphi \rightarrow \psi$ holds and φ holds, then also ψ must hold.

  In particular, if $\varphi \rightarrow \psi$ is a tautology and φ holds, we can deduce that also ψ holds.

  Example: $(p \wedge q) \rightarrow p$ is a tautology.

  If $p \wedge q$ holds, then also p holds.

  Example: $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$ is a tautology.

  If $p \rightarrow q$ holds and $\neg q$ holds, then $\neg p$ holds.

Tautologies

- If $\varphi \leftrightarrow \psi$ holds and φ holds, then ψ must hold.
  Similarly, if $\varphi \leftrightarrow \psi$ holds and ψ holds, then φ must hold.

  Example: $(\neg p \rightarrow q) \leftrightarrow (q \vee p)$ is a tautology.

  - If $\neg p \rightarrow q$ holds, then also $q \vee p$ must hold.
  - If $q \vee p$ holds, then also $\neg p \rightarrow q$ must hold.

Tautologies

When we take a tautology φ and replace all atomic propositions by arbitrary formulas, we obtain a tautology by this replacement.

Example: Formula $p \rightarrow (p \vee q)$ is a tautology.

This means that

$$\psi \rightarrow (\psi \vee \chi)$$

is a tautology for arbitrary formulas ψ and χ.

Replacement of atomic propositions:

- p is replaced with $q \vee \neg(r \rightarrow \neg s)$
- q is replaced with $\neg\neg(q \leftrightarrow p)$

We obtain tautology

$$(q \vee \neg(r \rightarrow \neg s)) \rightarrow ((q \vee \neg(r \rightarrow \neg s)) \vee \neg\neg(q \leftrightarrow p))$$

Contradictions

Definition

A formula φ is a **contradiction** if $v \not\models \varphi$ holds for every truth valuation v (i.e., when φ is false in every valuation).

Example: *"It is Wednesday today, and it is not Wednesday today."*

$$p \wedge \neg p$$

- φ is a tautology iff $\neg\varphi$ is a contradiction
- φ is a contradiction iff $\neg\varphi$ is a tautology

Satisfiable Formulas

Definition

A formula φ is **satisfiable** if there is at least one truth valuation v such that $v \models \varphi$.

- A formula is satisfiable iff it is not a contradiction.
- Every tautology is satisfiable but not every satisfiable formula is a tautology.

Example: A formula, which is satisfiable but not a tautology:

$$(p \vee q) \rightarrow p$$

- For example in valuation v_1, where $v_1(p) = 1$ and $v_1(q) = 0$, the formula is true.
- In valuation v_2, where $v_2(p) = 0$ and $v_2(q) = 1$, it is false.

Satisfiable Formulas

- φ is a tautology iff $\neg\varphi$ is not satisfiable
- φ is satisfiable iff $\neg\varphi$ is not a tautology

- **Satisfiable formulas**:
  - To show that a formula **is** satisfiable, it is sufficient to find a valuation, in which the formula is true.
  - To show that a formula **is not** satisfiable, it necessary to show that there is no valuation, in which the formula is true.

Tautologies and Contradictions

- **Tautologies**:
  - To show that a formula **is not** a tautology, it is sufficient to find a valuation, in which the formula is false.
  - To show that a formula **is** a tautology, it necessary to show that there is no valuation, in which the formula is false.
- **Contradictions**:
  - To show that formula **is not** a contradiction, it is sufficient to find a valuation, in which the formula is true.
  - To show that a formula **is** a contradiction, it necessary to show that there is no valuation, in which the formula is true.

Truth Valuations

For decing whether a formula φ is or is not a tautology (resp. a contradiction, satisfiable), the **table method** can be used:

- To go through all possible truth valuations systematically.

It is usually not necessary to construct the whole table. It is sufficient to concentrate on "interesting" cases.

- We can draw a graph representing the given formula and try to assign values 0 and 1 to its nodes in such a way that either we find an example of a truth valuation we are looking for (e.g., some valuation where the formula is false), or we find out that such valuation does not exist.

Truth Valuations

For example, for deciding if a formula is a tautology:

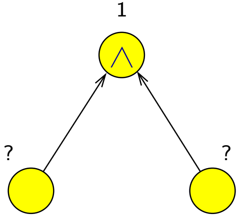
- We need to find out whether there exists some valuation where the formula is false.
- In this valuation, the node corresponding to the whole formula should have value 0.
- So we try to assign value 0 to this node.
- Then we try to assign values to other nodes in such a way that the assigned values are consistent with values assigned previously.
- If we succeed in labelling whole graph consistently, we have a valuation, in which the formula is false.

  In this case, it is clear that the formula is not a tautology.

Truth Valuations

- If some values were already assigned to some nodes, this assignment can impose some constraints on values that can be assigned to other nodes.
- Examples where some previously assigned values enforce some particular value at some other node (resp. nodes):

φ	ψ	$\varphi \wedge \psi$
0	0	0
0	1	0
1	0	0
1	1	1



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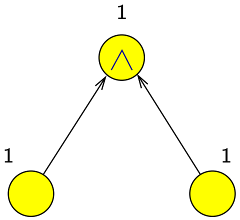

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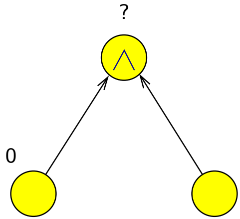

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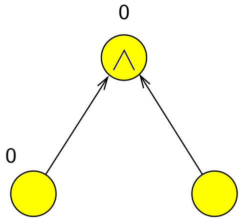

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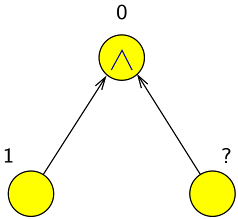

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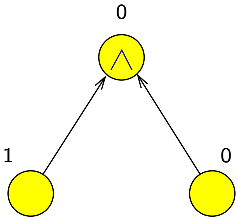

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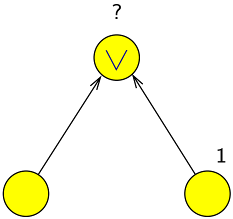

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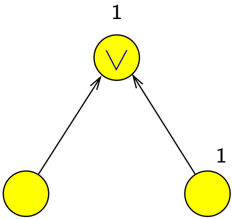

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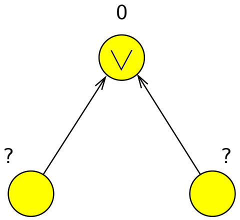

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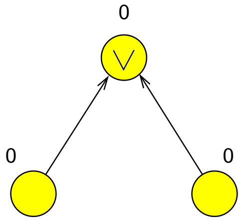

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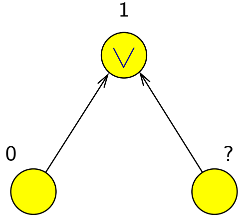

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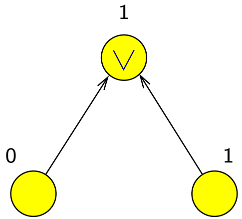

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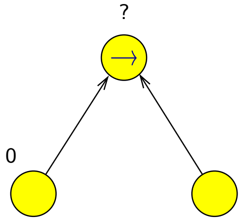

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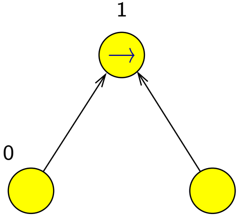

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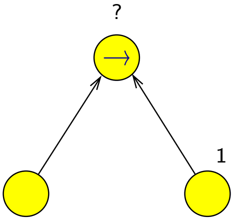

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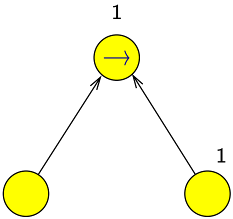

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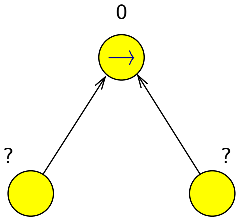

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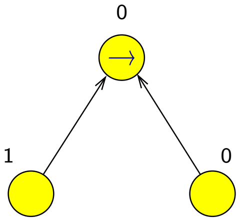

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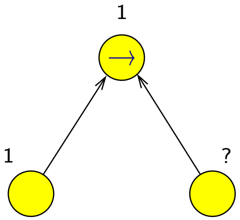

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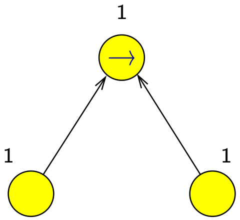

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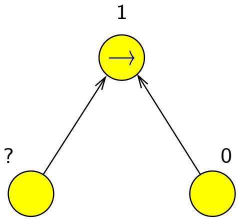

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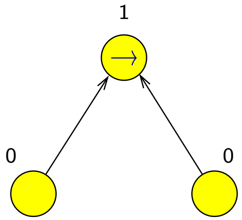

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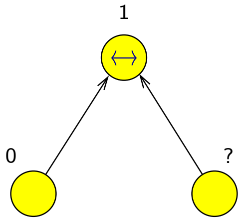

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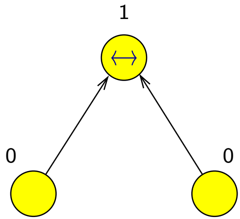

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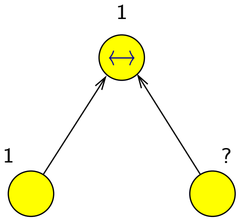

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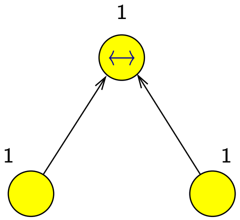

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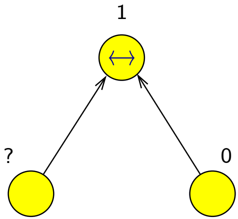

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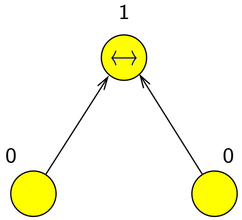

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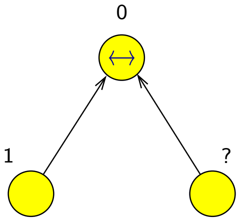

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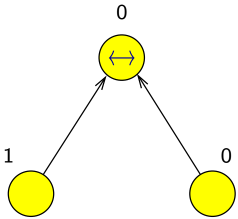
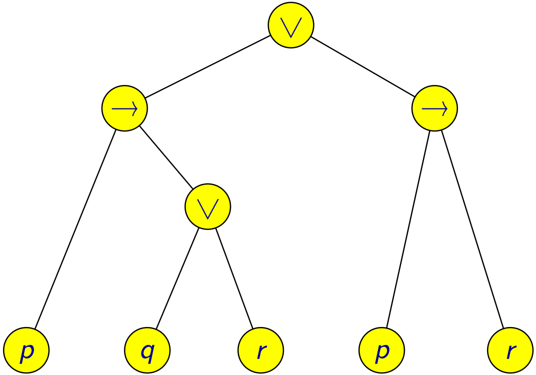


Is a given formula a tautology?

Example: $\varphi_1 := (p \rightarrow (q \vee r)) \vee (p \rightarrow r)$



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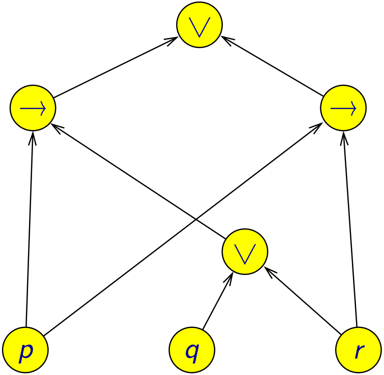

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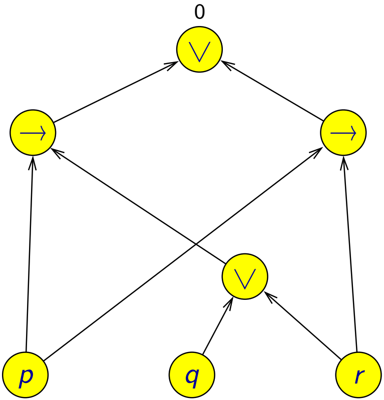

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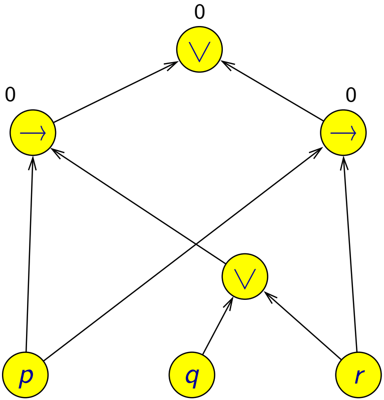

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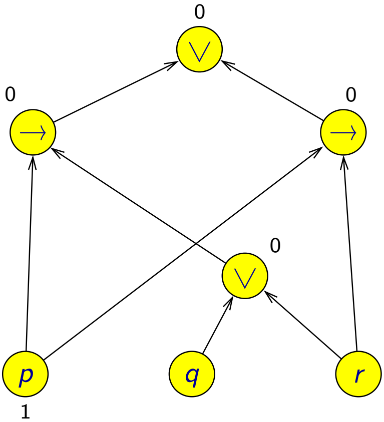

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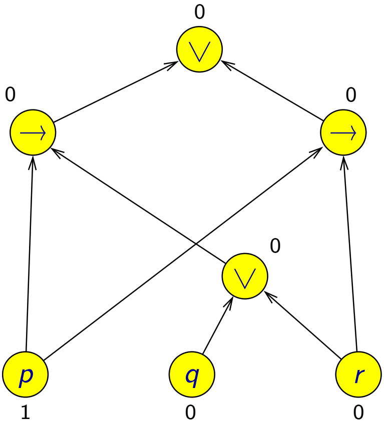

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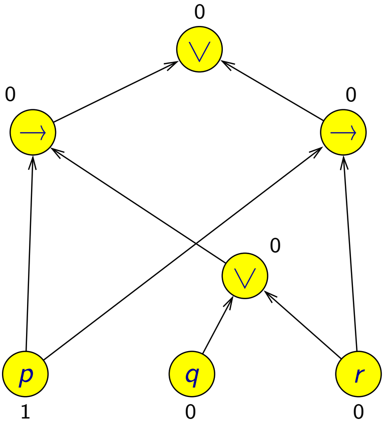
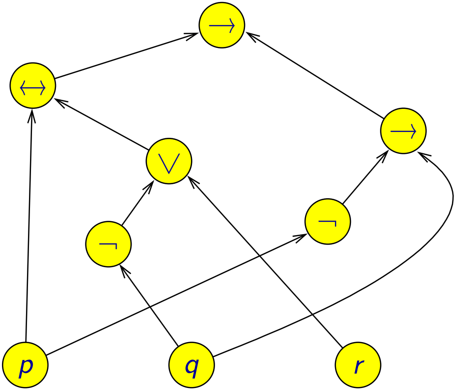


Formula φ_1 is not a tautology — it is false in valuation v where $v(p) = 1$, $v(q) = 0$, $v(r) = 0$.

Is a given formula a tautology?

Example: $\varphi_2 \;:=\; (p \leftrightarrow (\neg q \vee r)) \rightarrow (\neg p \rightarrow q)$



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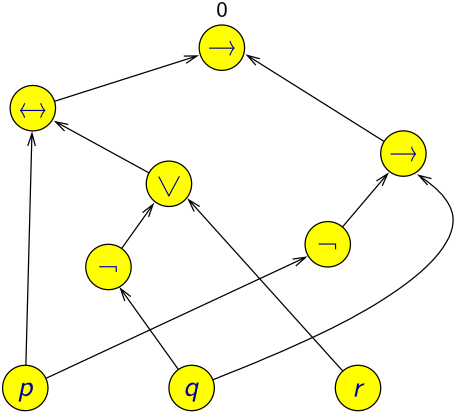

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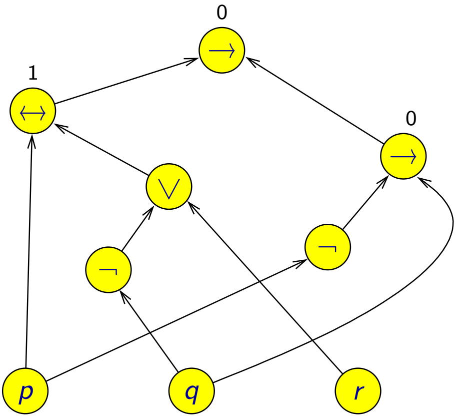

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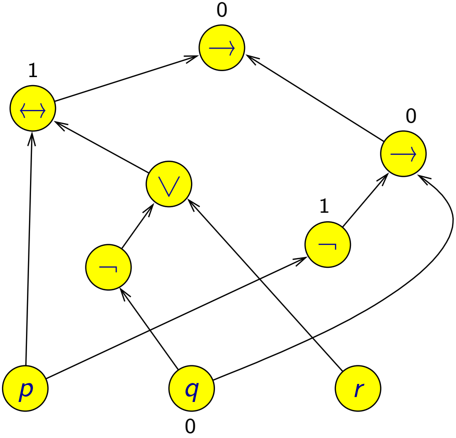

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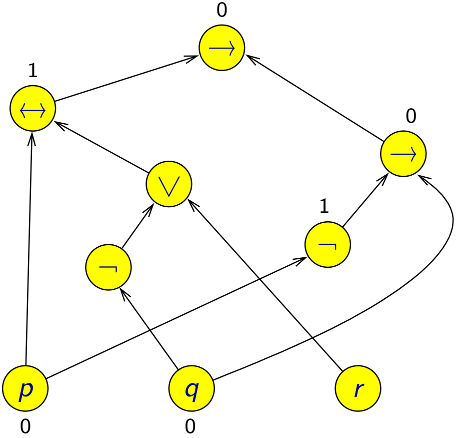

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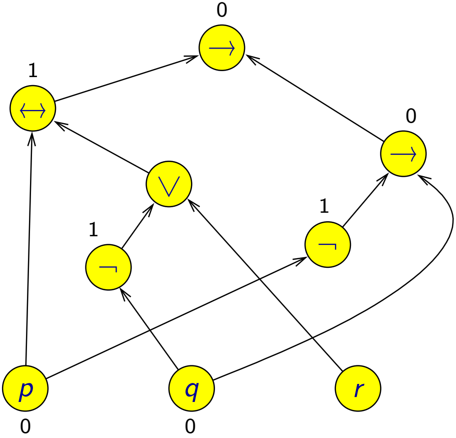

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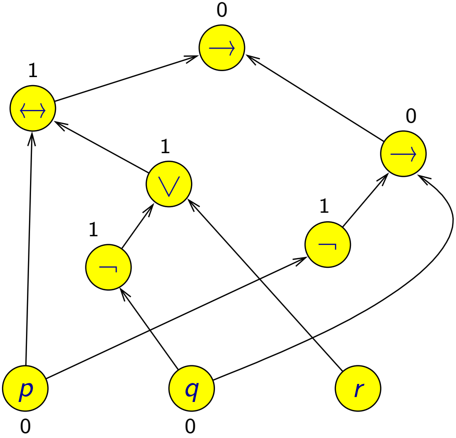

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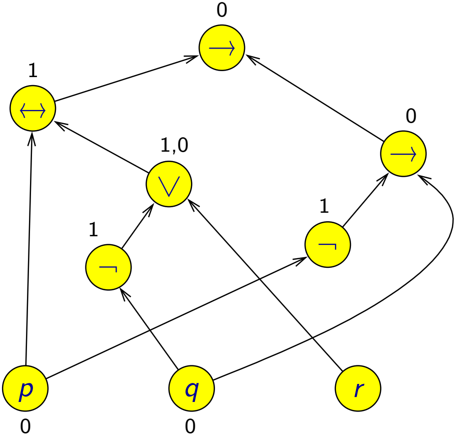

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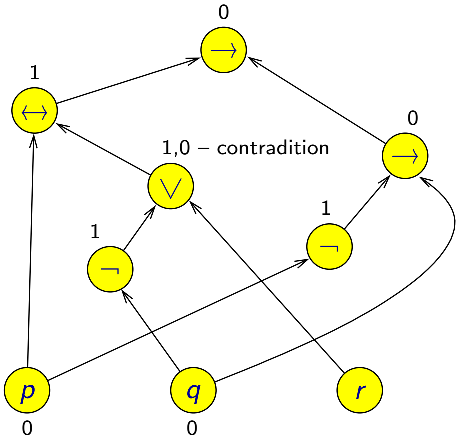

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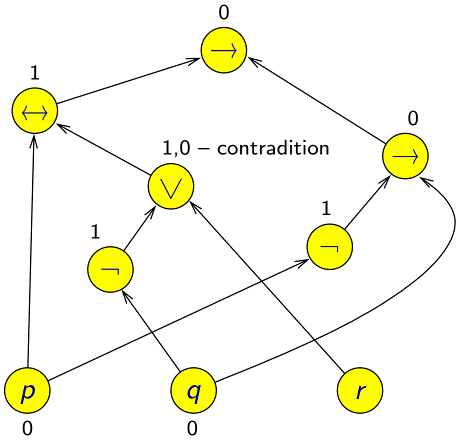


Formula φ_2 is a tautology.

Semantic Contradiction

Semantic contradition — the case when we find out that in a valuation with the given property we are looking for (e.g., a valuation where a given formula is false), some formula should be true and false at the same time.

- There could not exist a valuation where some formula would be true and false at the same time.
- This way we can justify for example that a given formula is a tautology (and so always true), because by finding a semantic contradiction we show there can not exist a valuation where this formula is false.

Is a given formula a tautology?

The approach from the previous example can described by the following sequnce of arguments:

1.	Let us assume that $(p \leftrightarrow (\neg q \vee r)) \rightarrow (\neg p \rightarrow q)$ is false. Then:	
2.	$p \leftrightarrow (\neg q \vee r)$ is true	- it follows from 1.
3.	$\neg p \rightarrow q$ is false	- it follows from 1.
4.	$\neg p$ is true	- it follows from 3.
5.	q is false	- it follows from 3.
6.	p is false	- it follows from 4.
7.	$\neg q$ is true	- it follows from 5.
8.	$\neg q \vee r$ is true	- it follows from 7.
9.	$\neg q \vee r$ is false	- it follows from 2. a 6.
10.	It is not possible that $(p \leftrightarrow (\neg q \vee r)) \rightarrow (\neg p \rightarrow q)$ is false because if it would be so, $\neg q \vee r$ would have to be true and false at the same time in this case (see 8. a 9.).	

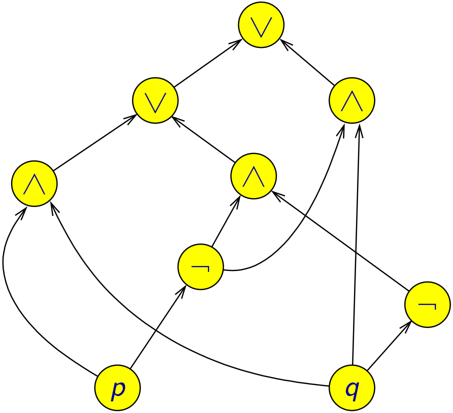
Remark: Note that in this justification the graph representing the given formula is not mentioned at all. We talk there only about truth and falsity of subformulas of this formula.

Is a given formula a tautology?

- It is not always the case that values that could be assigned to some nodes are uniquely determined by values previously assigned to some other nodes.
- When we are in a situation where it is not possible to assign a unique value to a node, it is necessary to try several possibilities.
- We choose some node and a value assigned to it. Then possibly some values that must be assigned to some other nodes are determined.
- If we do not succeed in finding a valuation we are looking for, we must backtrack, assign a diffent value to the given node, and try this new possibility.

Is a given formula a tautology?

Example: $\varphi_3 := ((p \wedge q) \vee (\neg p \wedge \neg q)) \vee (\neg p \wedge q)$



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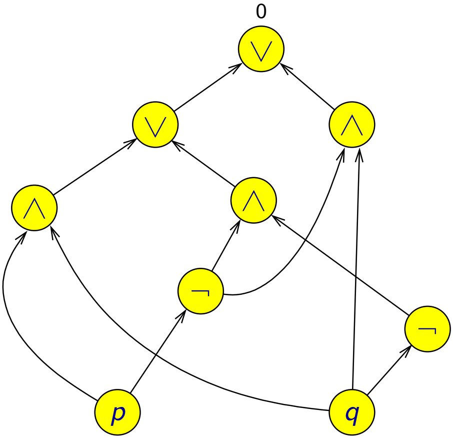

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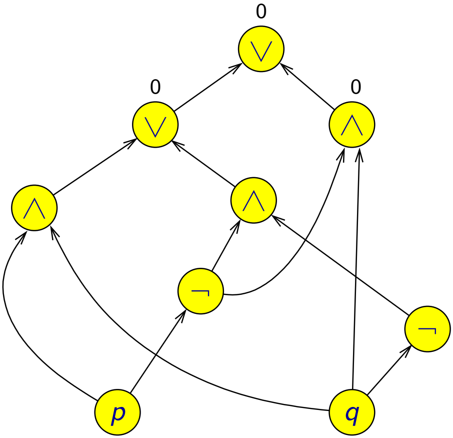

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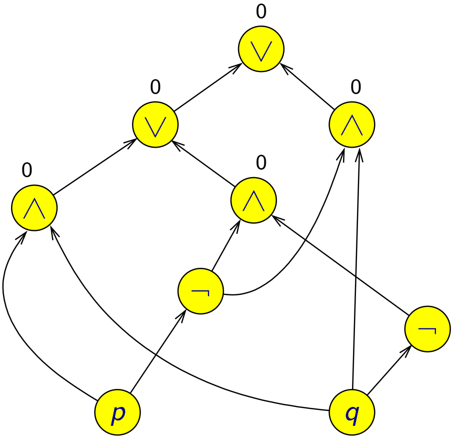

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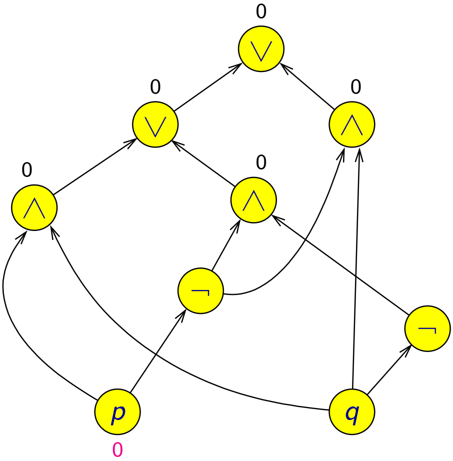

We try to assign value zero to node p.

Is a given formula a tautology?

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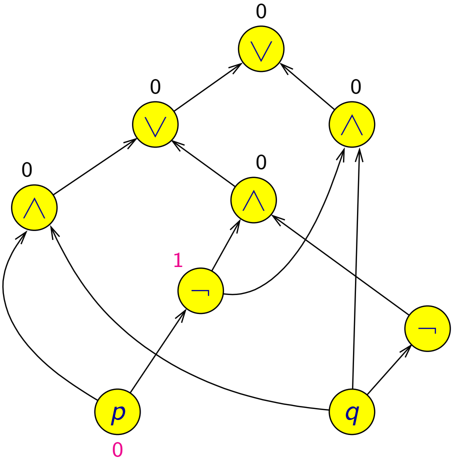

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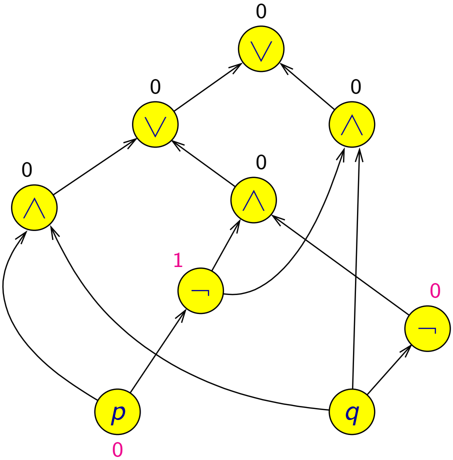

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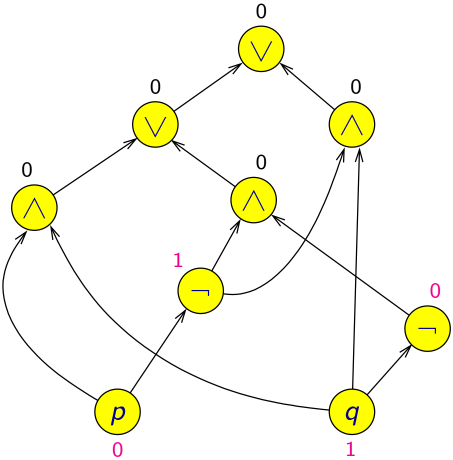

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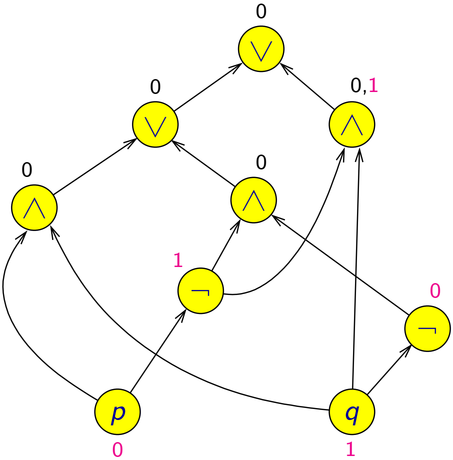

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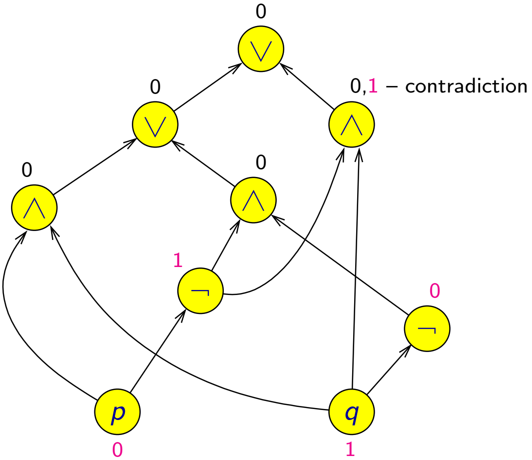

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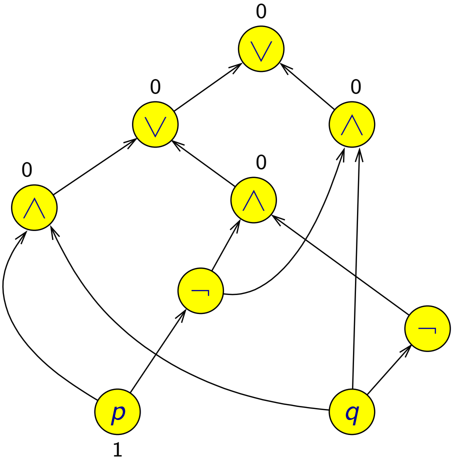

So node p must have value 1.

Is a given formula a tautology?

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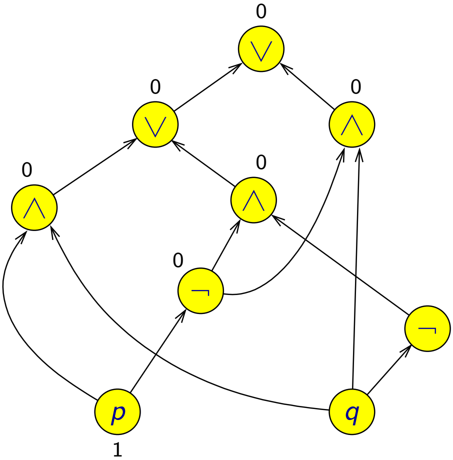

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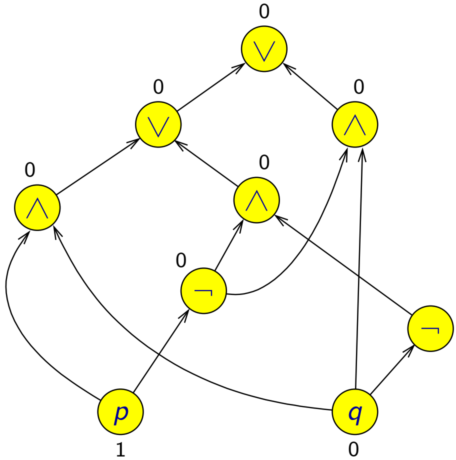

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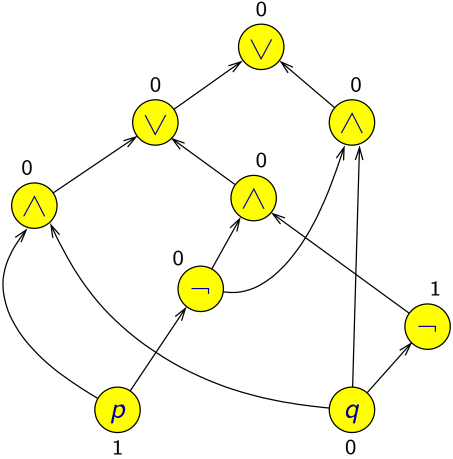

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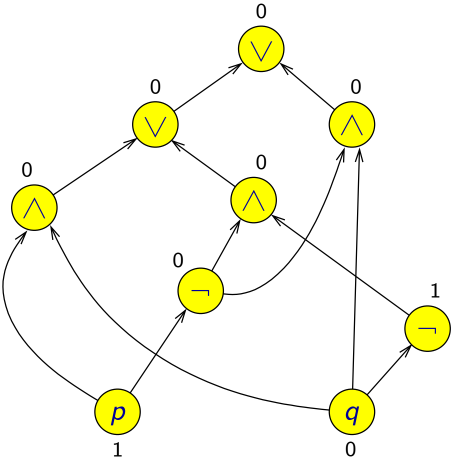

Formula φ_3 is not a tautology — it is false in valuation v where $v(p) = 1$, $v(q) = 0$.

Equivalence of Formulas

Definition

Formulas φ and ψ are **logically equivalent** if for each truth valuation v it holds that φ and ψ have the same truth value in valuation v, i.e.,

$$v \models \varphi \qquad \text{iff} \qquad v \models \psi.$$

The fact that formulas φ and ψ are logically equivalent is denoted

$$\varphi \;\Leftrightarrow\; \psi.$$

Formulas φ and ψ are logically equivalent iff $\varphi \leftrightarrow \psi$ is a tautology.

Equivalence of Formulas

Example: $\neg(p \rightarrow q) \;\Leftrightarrow\; p \wedge \neg q$

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg q$	$p \wedge \neg q$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

Equivalence of Formulas

To show that formulas φ and ψ are **not** equivalent, it is sufficient to find a valuation v such that:

- $v \models \varphi$ and $v \not\models \psi$, or
- $v \not\models \varphi$ and $v \models \psi$.

Example: $p \vee (q \wedge r)$ is not equivalent to $(p \vee q) \wedge r$

Valuation v, where:

- $v(p) = 1$
- $v(q) = 1$
- $v(r) = 0$

In this valuation, $p \vee (q \wedge r)$ holds but $(p \vee q) \wedge r$ does not hold.

Some Important Equivalences

- Equivalences for negation:

$\neg\neg p \;\Leftrightarrow\; p$ — *double negation*

- Equivalences for conjunction:

$(p \wedge q) \wedge r \;\Leftrightarrow\; p \wedge (q \wedge r)$ — *associativity*

$p \wedge q \;\Leftrightarrow\; q \wedge p$ — *commutativity*

$p \wedge p \;\Leftrightarrow\; p$ — *idempotence*

- Equivalences for disjunction:

$(p \vee q) \vee r \;\Leftrightarrow\; p \vee (q \vee r)$ — *associativity*

$p \vee q \;\Leftrightarrow\; q \vee p$ — *commutativity*

$p \vee p \;\Leftrightarrow\; p$ — *idempotence*

Some Important Equivalences

– Distributivity of $\wedge$ and $\vee$:

$$p \wedge (q \vee r) \;\Leftrightarrow\; (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \;\Leftrightarrow\; (p \vee q) \wedge (p \vee r)$$

– De Morgan's laws:

$$\neg(p \wedge q) \;\Leftrightarrow\; \neg p \vee \neg q$$
$$\neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q$$

– Equivalences for implication:

$$p \rightarrow q \;\Leftrightarrow\; \neg p \vee q$$
$$\neg(p \rightarrow q) \;\Leftrightarrow\; p \wedge \neg q$$

Some Important Equivalences

– Equivalences for $\leftrightarrow$:

$(p \leftrightarrow q) \leftrightarrow r \;\Leftrightarrow\; p \leftrightarrow (q \leftrightarrow r)$	*associativity*
$p \leftrightarrow q \;\Leftrightarrow\; q \leftrightarrow p$	*commutativity*
$p \leftrightarrow q \;\Leftrightarrow\; (p \rightarrow q) \wedge (q \rightarrow p)$	
$p \leftrightarrow q \;\Leftrightarrow\; (p \vee \neg q) \wedge (\neg p \vee q)$	
$p \leftrightarrow q \;\Leftrightarrow\; (p \wedge q) \vee (\neg p \wedge \neg q)$	

Equivalence of Formulas

Let us assume that formulas φ and ψ are logically equivalent, i.e.,

$$\varphi \Leftrightarrow \psi .$$

If we replace atomic propositions in φ and ψ by arbitrary formulas, we obtain again a pair of equivalent formulas.

Example: $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Therefore, for arbitrary formulas χ_1 and χ_2 is

$$\neg(\chi_1 \vee \chi_2) \Leftrightarrow \neg\chi_1 \wedge \neg\chi_2$$

Equivalence of Formulas

$$\neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q$$

Replacement of atomic propositions:

- p replaced by $q \vee \neg(r \rightarrow \neg s)$
- q replaced by $\neg(q \leftrightarrow p)$

We obtain

$$\neg((q \vee \neg(r \rightarrow \neg s)) \vee \neg(q \leftrightarrow p)) \;\Leftrightarrow\; \neg(q \vee \neg(r \rightarrow \neg s)) \wedge \neg\neg(q \leftrightarrow p)$$

Equivalence of Formulas

Let us assume that φ is a formula and ψ its subformula.

If we replace some occurrence of subformula ψ in formula φ with a formula ψ' such that $\psi \Leftrightarrow \psi'$, we obtain a formula φ' such that

$$\varphi \Leftrightarrow \varphi' .$$

Example: In formula

$$\neg((p \rightarrow q) \vee (\neg(p \rightarrow q) \rightarrow r))$$

we replace the second occurrence of subformula $p \rightarrow q$ with an equivalent formula $\neg p \vee q$.

We obtain

$$\neg((p \rightarrow q) \vee (\neg(\neg p \vee q) \rightarrow r))$$

Equivalent Transformations

For arbitrary formulas φ, ψ, and χ, it holds:

- $\varphi \Leftrightarrow \varphi$.
- If $\varphi \Leftrightarrow \psi$, then $\psi \Leftrightarrow \varphi$.
- If $\varphi \Leftrightarrow \psi$ and $\psi \Leftrightarrow \chi$, then $\varphi \Leftrightarrow \chi$.

When we try to prove equivalence of formulas, we can proceed by smaller steps:

For example, if it holds that $\varphi_1 \Leftrightarrow \varphi_2$, $\varphi_2 \Leftrightarrow \varphi_3$, $\varphi_3 \Leftrightarrow \varphi_4$, and $\varphi_4 \Leftrightarrow \varphi_5$, we can conclude that

$$\varphi_1 \Leftrightarrow \varphi_5 .$$

This can be written as

$$\varphi_1 \Leftrightarrow \varphi_2 \Leftrightarrow \varphi_3 \Leftrightarrow \varphi_4 \Leftrightarrow \varphi_5$$

Equivalent Transformations

Example: The proof that

$$(p \wedge q) \rightarrow r \;\Leftrightarrow\; p \rightarrow (q \rightarrow r)$$

$$\begin{aligned}
(p \wedge q) \rightarrow r \;&\Leftrightarrow\; \neg(p \wedge q) \vee r \\
&\Leftrightarrow\; (\neg p \vee \neg q) \vee r \\
&\Leftrightarrow\; \neg p \vee (\neg q \vee r) \\
&\Leftrightarrow\; \neg p \vee (q \rightarrow r) \\
&\Leftrightarrow\; p \rightarrow (q \rightarrow r)
\end{aligned}$$

Equivalent Transformations

Every formula can be transformed to an equivalent formula that uses only "$\neg$", "$\wedge$", and "$\vee$" as logical connectives.

- The connective "$\leftrightarrow$" can be replaced by other connectives using the following equivalences:
  - $p \leftrightarrow q \;\Leftrightarrow\; (p \rightarrow q) \wedge (q \rightarrow p)$
  - $p \leftrightarrow q \;\Leftrightarrow\; (p \vee \neg q) \wedge (\neg p \vee q)$
  - $p \leftrightarrow q \;\Leftrightarrow\; (p \wedge q) \vee (\neg p \wedge \neg q)$
- The connective "$\rightarrow$" can be replaced by "$\neg$" and "$\vee$" using the following equivalence:
  - $p \rightarrow q \;\Leftrightarrow\; \neg p \vee q$

Example:

$$\begin{aligned}(\neg q \rightarrow r) \wedge \neg(p \leftrightarrow r) \;&\Leftrightarrow\; (\neg\neg q \vee r) \wedge \neg(p \leftrightarrow r)\\ &\Leftrightarrow\; (\neg\neg q \vee r) \wedge \neg((p \wedge r) \vee (\neg p \wedge \neg r))\end{aligned}$$

Equivalent Transformations

Every formula can be transformed to an equivalent formula, which contains only logical connectives "$\neg$", "$\wedge$" and "$\vee$", and where negations are applied only to atomic propositions.

- We can assume that formula contains only "$\neg$", "$\wedge$" and "$\vee$".
- Negations can be "pushed" to atomic propositions using the following equivalences:
  - $\neg\neg p \;\Leftrightarrow\; p$
  - $\neg(p \wedge q) \;\Leftrightarrow\; \neg p \vee \neg q$
  - $\neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q$

Example:

$$
\begin{aligned}
& (\neg\neg q \vee r) \wedge \neg((p \wedge r) \vee (\neg p \wedge \neg r)) \\
& \qquad \Leftrightarrow\ (q \vee r) \wedge \neg((p \wedge r) \vee (\neg p \wedge \neg r)) \\
& \qquad \Leftrightarrow\ (q \vee r) \wedge (\neg(p \wedge r) \wedge \neg(\neg p \wedge \neg r)) \\
& \qquad \Leftrightarrow\ (q \vee r) \wedge ((\neg p \vee \neg r) \wedge \neg(\neg p \wedge \neg r)) \\
& \qquad \Leftrightarrow\ (q \vee r) \wedge ((\neg p \vee \neg r) \wedge (\neg\neg p \vee \neg\neg r)) \\
& \qquad \Leftrightarrow\ (q \vee r) \wedge ((\neg p \vee \neg r) \wedge (p \vee \neg\neg r)) \\
& \qquad \Leftrightarrow\ (q \vee r) \wedge ((\neg p \vee \neg r) \wedge (p \vee r))
\end{aligned}
$$

Logical Constants

For some purposes it can be useful to introduce the following special formulas:

- $\top$ — a formula, which is always true
- $\bot$ — a formula, which is always false

For every truth valuation v it holds:

- $v \models \top$ ($\top$ has always truth value 1)
- $v \not\models \bot$ ($\bot$ has always truth value 0)

Logical Constants

Symbols $\top$ and $\bot$ can be viewed as abbreviations:

- $\top$ stands for an arbitrary tautology (e.g., $p \rightarrow p$)
- $\bot$ stands for an arbitrary contradiction (e.g., $p \wedge \neg p$)

Alternatively, we could extend the definition of syntax and semantics of propositional logic.

Symbols $\top$ and $\bot$ can be viewed as logical connectives with arity 0.

Logical Constants

Examples of equivalences that hold for $\top$ and $\bot$ (and for arbitrary p):

$$\begin{aligned} \top &\Leftrightarrow p \vee \neg p \\ \neg\top &\Leftrightarrow \bot \\ p \wedge \top &\Leftrightarrow p \\ p \vee \top &\Leftrightarrow \top \end{aligned} \qquad\qquad \begin{aligned} \bot &\Leftrightarrow p \wedge \neg p \\ \neg\bot &\Leftrightarrow \top \\ p \vee \bot &\Leftrightarrow p \\ p \wedge \bot &\Leftrightarrow \bot \end{aligned}$$

Equivalence of Formulas

It is **not** necessary for equivalent formulas to contain the same atomic propositions.

Example: $(q \rightarrow \neg\neg q) \wedge \neg p \;\Leftrightarrow\; p \rightarrow (r \wedge \neg r)$

$$
\begin{aligned}
(q \rightarrow \neg\neg q) \wedge \neg p &\Leftrightarrow (q \rightarrow q) \wedge \neg p \\
&\Leftrightarrow \top \wedge \neg p \\
&\Leftrightarrow \neg p \\
&\Leftrightarrow \neg p \vee \bot \\
&\Leftrightarrow p \rightarrow \bot \\
&\Leftrightarrow p \rightarrow (r \wedge \neg r)
\end{aligned}
$$

For example, also all tautologies are logically equivalent.

Conjunctions and Disjunctions of Several Formulas

Due to associativity of conjunction, it holds for example:

$$p \wedge ((q \wedge r) \wedge (s \wedge t)) \;\Leftrightarrow\; (p \wedge q) \wedge ((r \wedge s) \wedge t)$$

Both these formulas are also equivalent to formulas

- $p \wedge (q \wedge (r \wedge (s \wedge t)))$
- $(((p \wedge q) \wedge r) \wedge s) \wedge t$

All these formulas are true iff all propositions p, q, r, s, and t are true.

Convention: Due to associativity of conjunction, the parentheses can be omitted and we can write

$$p \wedge q \wedge r \wedge s \wedge t$$

Conjunctions and Disjunctions of Several Formulas

Because conjunction is not only associative but also commutative, the order of members of such more complicated conjunction is not important. For example:

$$r \wedge t \wedge q \wedge s \wedge p \quad \Leftrightarrow \quad p \wedge q \wedge r \wedge s \wedge t$$

Due to idempotence, also the number of occurrences of each member is not important.
For example:

$$p \wedge q \wedge p \quad \Leftrightarrow \quad q \wedge p \wedge q \wedge q$$

Conjunctions and Disjunctions of Several Formulas

The same holds also for disjunction, e.g.:

$$(p \vee q) \vee (r \vee q) \quad \Leftrightarrow \quad q \vee (p \vee (r \vee (r \vee r)))$$

Convention: Instead of $(p \vee q) \vee (r \vee (s \vee t))$ we can write

$$p \vee q \vee r \vee s \vee t$$

All this holds not only for atomic propositions but also for arbitrary formulas, e.g.:

- Instead of $(\varphi_1 \wedge \varphi_2) \wedge (\varphi_3 \wedge (\varphi_4 \wedge \varphi_5))$ we can write

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_5$$

Conjunctions and Disjunctions of Several Formulas

Conjunction of n formulas φ_1, φ_2, ..., φ_n, where $n \geq 0$, is the formula

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$$

In particular:

- For $n = 0$, the conjunction is the formula $\top$.
- For $n = 1$, the conjunction is the formula φ_1.

Disjunction of n formulas φ_1, φ_2, ..., φ_n, where $n \geq 0$, is the formula

$$\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$$

In particular:

- For $n = 0$, the disjunction is the formula $\perp$.
- For $n = 1$, the disjunction is the formula φ_1.

Conjunctions and Disjunctions of Several Formulas

Conjunction $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$:

- The whole formula is true iff all formulas φ_1, φ_2, ..., φ_n are true.
- If some formula φ_i is equivalent to $\perp$, then the whole formula is equivalent to $\perp$.
- If some formula φ_i is equivalent to a negation of some formula φ_j (i.e., $\varphi_i \Leftrightarrow \neg\varphi_j$), then the whole formula is equivalent to $\perp$.
- If some formula φ_i is equivalent to $\top$, then it is possible to omit the formula φ_i from the whole formula.

Conjunctions and Disjunctions of Several Formulas

Disjunction $\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$:

- The whole formula is true iff at least one of formulas φ_1, φ_2, ..., φ_n is true.
- If some formula φ_i is equivalent to $\top$, then the whole formula is equivalent to $\top$.
- If some formula φ_i is equivalent to a negation of some formula φ_j (i.e., $\varphi_i \Leftrightarrow \neg\varphi_j$), then the whole formula is equivalent to $\top$.
- If some formula φ_i is equilvalent to $\bot$, then it is possible to omit the formula φ_i from the whole formula.

Normal Forms of Fomulas

- **Literal** — an atomic proposition or its negation, e.g.,

$$p \qquad\qquad \neg q \qquad\qquad \neg r$$

- An **elementary conjunction** — a conjunction of one or more literals, e.g.,

$$(p \wedge \neg q) \qquad\qquad (r) \qquad\qquad (q \wedge \neg r \wedge p)$$

- An **elementary disjunction** (**clause**) — a disjunction of one or more literals, e.g.,

$$(p \vee \neg q) \qquad\qquad (r) \qquad\qquad (q \vee \neg r \vee p)$$

Example:

- Elementary conjunction

$$(p \wedge \neg q \wedge r \wedge \neg s \wedge \neg t)$$

is **true** in exactly those truth valuations v where

$$v(p) = 1 \qquad v(q) = 0 \qquad v(r) = 1 \qquad v(s) = 0 \qquad v(t) = 0$$

- Elementary disjunction

$$(p \vee \neg q \vee r \vee \neg s \vee \neg t)$$

is **false** in exactly those truth valuations v where

$$v(p) = 0 \qquad v(q) = 1 \qquad v(r) = 0 \qquad v(s) = 1 \qquad v(t) = 1$$

Normal Forms of Fomulas

- **Disjunctive normal form** (**DNF**) — a disjunction of zero or more elementary conjunctions, e.g.,

$$(p \wedge \neg q) \ \vee \ (\neg r) \ \vee \ (\neg r \wedge \neg p \wedge \neg q)$$

- **Conjunctive normal form** (**CNF**) — a conjunction of zero or more elementary disjunctions (clauses), e.g.,

$$(p \vee \neg q) \ \wedge \ (\neg r) \ \wedge \ (\neg r \vee \neg p \vee \neg q)$$

Remark: So formula $\perp$ is a special case of a formula in DNF, and formula $\top$ is a special case of a formula in CNF.

Normal Forms of Fomulas

A formula in CNF is a **tautology** iff for each elemetary disjunction there is some atomic proposition p such that literals p and $\neg p$ occur in the elemetary disjunction.

Example: $(p \vee q \vee \neg r \vee \neg q) \wedge (\neg p \vee \neg s \vee s) \wedge (t \vee \neg r \vee s \vee \neg t \vee q)$

A formula in DNF is a **contradiction** iff for each elemetary conjunction there is some atomic proposition p such that literals p and $\neg p$ occur in the elemetary conjunction.

Example: $(p \wedge q \wedge \neg r \wedge \neg q) \vee (\neg p \wedge \neg s \wedge s) \vee (t \wedge \neg r \wedge s \wedge \neg t \vee q)$

Normal Forms of Fomulas

Transformation of a formula to DNF and CNF:

- We can assume that the formula contains only atomic propositions, connectives "$\neg$" applied to atomic propositions, and connectives "$\bigwedge$" and "$\bigvee$".
- The required form of the formula can be obtained by use of the following equivalences:
  - $p \wedge (q \vee r) \;\Leftrightarrow\; (p \wedge q) \vee (p \wedge r)$ — for transformation to DNF
  - $p \vee (q \wedge r) \;\Leftrightarrow\; (p \vee q) \wedge (p \vee r)$ — for transformation to CNF

Normal Forms of Fomulas

Example: Transformation of formula $q \wedge ((\neg p \vee \neg r) \wedge (p \vee r))$ to DNF:

$$
\begin{aligned}
& q \wedge ((\neg p \vee \neg r) \wedge (p \vee r)) \\
\Leftrightarrow\ & (q \wedge (\neg p \vee \neg r)) \wedge (p \vee r) \\
\Leftrightarrow\ & ((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge (p \vee r) \\
\Leftrightarrow\ & (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge p) \ \vee\ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (((q \wedge \neg p) \wedge p) \vee ((q \wedge \neg r) \wedge p)) \ \vee\ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (q \wedge \neg p \wedge p) \ \vee\ (q \wedge \neg r \wedge p) \ \vee\ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (q \wedge \bot) \ \vee\ (q \wedge \neg r \wedge p) \ \vee\ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & \bot \ \vee\ (q \wedge \neg r \wedge p) \ \vee\ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (q \wedge \neg r \wedge p) \ \vee\ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & (p \wedge q \wedge \neg r) \ \vee\ (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
\Leftrightarrow\ & \quad \ldots
\end{aligned}
$$

Normal Forms of Fomulas

$$
\begin{aligned}
&\dots \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (((q \wedge \neg p) \vee (q \wedge \neg r)) \wedge r) \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (((q \wedge \neg p) \wedge r) \vee ((q \wedge \neg r) \wedge r)) \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (q \wedge \neg p \wedge r) \vee (q \wedge \neg r \wedge r) \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (q \wedge \neg r \wedge r) \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (q \wedge \bot) \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee \bot \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)
\end{aligned}
$$

Normal Forms of Fomulas

It is easy to construct a formula in CNF or DNF for a given truth table:

p	q	r	φ
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

DNF:
$(\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r)$

CNF:
$(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

Normal Forms of Fomulas

When we consider a fixed **finite** set of atomic propositions At:

- **Complete disjunctive normal form** (**CDNF**) — a formula in DNF, where every elementary conjunction contains every atomic proposition from At exactly once.

  Example: $(p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)$

- **Complete conjunctive normal form** (**CCNF**) — a formula in CNF, where every clause contains every atomic proposition from At exactly once.

  Example: $(p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$

Remark: In the examples is $At = \{p, q, r\}$.

Minimal Sets of Logical Connectives

We can see from the previous discussion that connectives "$\neg$", "$\wedge$", and "$\vee$" suffice for contructing a formula for every truth table.

In fact, some smaller sets of logical connectives are sufficient for this purpose:

- "$\neg$", "$\wedge$":
  $\varphi \vee \psi$ can be expressed as $\neg(\neg\varphi \wedge \neg\psi)$
- "$\neg$", "$\vee$":
  $\varphi \wedge \psi$ can be expressed as $\neg(\neg\varphi \vee \neg\psi)$
- "$\neg$", "$\rightarrow$":
  $\varphi \vee \psi$ can be expressed as $\neg\varphi \rightarrow \psi$
  $\varphi \wedge \psi$ can be expressed as $\neg(\varphi \rightarrow \neg\psi)$

Minimal Sets of Logical Connectives

- "$\rightarrow$", "$\bot$":
  $\neg\varphi$ can be expressed as $\varphi \rightarrow \bot$
  $\varphi \vee \psi$ can be expressed as $(\varphi \rightarrow \bot) \rightarrow \psi$
  $\varphi \wedge \psi$ can be expressed as $(\varphi \rightarrow (\psi \rightarrow \bot)) \rightarrow \bot$

- "$|$" — NAND — Sheffer stroke (also denoted by "$\uparrow$"):

φ	ψ	$\varphi \mid \psi$
0	0	1
0	1	1
1	0	1
1	1	0

  $\neg\varphi$ can be expressed as $\varphi \mid \varphi$
  $\varphi \vee \psi$ can be expressed as $(\varphi \mid \varphi) \mid (\psi \mid \psi)$
  $\varphi \wedge \psi$ can be expressed as $(\varphi \mid \psi) \mid (\varphi \mid \psi)$

Minimal Sets of Logical Connectives

- "$\downarrow$" — NOR — Peirce's arrow:

φ	ψ	$\varphi \downarrow \psi$
0	0	1
0	1	0
1	0	0
1	1	0

$\neg\varphi$ can be expressed as $\varphi \downarrow \varphi$
$\varphi \vee \psi$ can be expressed as $(\varphi \downarrow \psi) \downarrow (\varphi \downarrow \psi)$
$\varphi \wedge \psi$ can be expressed as $(\varphi \downarrow \varphi) \downarrow (\psi \downarrow \psi)$